Ingredients of a Logical Analysis of Rational Agency

⇒ informational attitudes (eg., knowledge, belief, certainty)
⇒ time, actions and ability
⇒ motivational attitudes (eg., preferences)
⇒ group notions (eg., common knowledge and coalitional ability)
⇒ normative attitudes (eg., obligations)
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Time

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Many variations

- discrete or continuous
- branching or linear
- point based or interval based

See, for example,


Models of Time

\[ \mathcal{T} = \langle T, <, V \rangle \] where

- \( T \) is a set of **time points** (or **moments**),
- \( < \subseteq T \times T \) is the **precedence relation**: \( s < t \) means “time point \( s \) precedes time point \( t \) (or \( s \) occurs earlier than \( t \))” and
- \( V : At \rightarrow \wp(T) \) is a valuation function (describing when the atomic propositions are true).
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Examples: \( \langle \mathbb{N}, < \rangle, \langle \mathbb{Z}, < \rangle, \langle \mathbb{Q}, < \rangle, \langle \mathbb{R}, < \rangle \rangle \)
Other properties of $<$

- **Linearity**: for all $s, t \in T$, $s < t$ or $s = t$ of $t < s$

- **Past-linear**: for all $s, x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$

- **Denseness** for all $s, t \in T$, if $s < t$ then there is a $z \in T$ such that $s < z$ and $z < t$

- **Discreteness**: for all $s, t \in T$, if $s < t$ then there is a $z$ such that $(s < z$ and there is no $u$ such that $s < u$ and $u < z)$
Priorean Temporal Logic

Let $\mathcal{L}_t$ be defined by the following grammar

$$p \mid \neg \varphi \mid \varphi \land \psi \mid G\varphi \mid H\varphi$$

where $p \in \text{At}$. 
Priorean Temporal Logic

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\[ p \mid \neg \varphi \mid \varphi \land \psi \mid G\varphi \mid H\varphi \]

where $p \in \text{At}$.

\(G\varphi\): “\(\varphi\) is **going to** become true”

\(H\varphi\): “\(\varphi\) has been true”

\(F\varphi := \neg G\neg\varphi\): “\(\varphi\) is true in the future”

\(P\varphi := \neg H\neg\varphi\): “\(\varphi\) was true some time in the past”
\[ \mathcal{M} = \langle T, <, V \rangle \]

- \( \mathcal{M}, t \models p \) iff \( t \in V(p) \)
- \( \mathcal{M}, t \models \neg \varphi \) iff \( \mathcal{M}, t \not\models \varphi \)
- \( \mathcal{M}, t \models \varphi \land \psi \) iff \( \mathcal{M}, t \models \varphi \) and \( \mathcal{M}, t \models \psi \)
- \( \mathcal{M}, t \models G \varphi \) iff for all \( s \in T \), if \( t < s \) then \( \mathcal{M}, s \models \varphi \)
- \( \mathcal{M}, t \models H \varphi \) iff for all \( s \in T \), if \( s < t \) then \( \mathcal{M}, s \models \varphi \)

- \( \mathcal{M}, t \models F \varphi \) iff there is \( s \in T \) such that \( t < s \) and \( \mathcal{M}, s \models \varphi \)
- \( \mathcal{M}, t \models P \varphi \) iff there is \( s \in T \) such that \( s < t \) and \( \mathcal{M}, s \models \varphi \)
Frame Correspondence

- $H \perp \lor PH \perp$ is valid
Frame Correspondence

- $H \perp \lor PH \perp$ is valid iff there is a starting point
Frame Correspondence

- $H \perp \lor PH \perp$ is valid iff there is a starting point
- $P \top$ is valid
- $F \phi \rightarrow (P \phi \lor \phi \lor F \phi)$ is valid iff the future is not branching
- $F \phi \rightarrow FF \phi$ is valid iff the flow of time is dense
- $(F \top \land \phi \land H \phi) \rightarrow FH \phi$ is valid iff the flow of time is discrete
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**Logic and Artificial Intelligence 8/30**
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Basic Temporal Logic

All classical propositional tautologies

Distribution
\[ G(\varphi \to \psi) \to (G\varphi \to G\psi) \]
\[ G(\varphi \to \psi) \to (G\varphi \to G\psi) \]

Converse
\[ \varphi \to \GP \varphi \]
\[ \varphi \to \HF \varphi \]

Transitivity: \[ G\varphi \to GG\varphi \]

Modus Ponens: from \( \varphi \) and \( \varphi \to \psi \) infer \( \psi \)

Temporal Generalization: from \( \varphi \) infer \( F\varphi \); from \( \varphi \) infer \( G\varphi \)
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**Theorem.** The above logic is sound and complete with respect to the class of all flows of time
Theorem. The above logic with the linearity axioms is sound and complete with respect to the class of all linear flows of time

- $PF\varphi \rightarrow (P\varphi \lor \varphi \lor F\varphi)$
- $FP\varphi \rightarrow (F\varphi \lor \varphi \lor P\varphi)$
Other Languages: Since and Until

\[ M, t \models \varphi U \psi \text{ iff } M, s \models \psi \text{ for some } s \text{ such that } t < s \text{ and } \]
\[ M, u \models \varphi \text{ for all } u \text{ with } t < u < s \]

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\textbf{Theorem} (Kamp). Over the class of linear, continuous orderings, every temporal operator can be defined using the above modalities.
Branching Time

Each moment $t \in T$ can be decided into the $Past(t) = \{s \in T \mid s < t\}$ and the $Future(t) = \{s \in T \mid t < s\}$ ("A-series")

Typically, it is assumed that the past is linear, but the future may be branching.
Branching Time

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\[ \text{Past}(t) = \{ s \in T \mid s < t \} \]
and the \( \text{Future}(t) = \{ s \in T \mid t < s \} \) (**"A-series"**) 

Typically, it is assumed that the past is linear, but the future may be branching.

\( F \varphi: \) “it will be the case that \( \varphi \)”

\( \varphi \) will be the case “in the case in the actual course of events” or “no matter what course of events”
Branching Time Logics

A branch $b$ in $\langle T, < \rangle$ is a maximal linearly ordered subset of $T$

$s \in T$ is on a branch $b$ of $T$ provided $s \in b$ (we also say “$b$ is a branch going through $t$”).
Branching Time Logics

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M, t, b \models p \text{ iff } t \in V(p) \\
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M, t, b \models H\varphi \text{ iff for all } s \in T, \text{ if } s \text{ is on } b \text{ and } s < t \text{ then } M, s, b \models \varphi \\
M, t, b \models \forall \varphi \text{ iff } M, s, c \models \varphi \text{ for all branches } c \text{ through } t
\]
Computational vs. Behavioral Structures

\[
x = 1
\]

\[
x = 2
\]
Temporal Logics

Linear Time Temporal Logic: Reasoning about computation paths:
$F \phi$: $\phi$ is true some time in the future.


Branching Time Temporal Logic: Allows quantification over paths:
$\exists F \phi$: there is a path in which $\phi$ is eventually true.

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\[ \exists Fp_{x=2} \]
Temporal Logics

\[ x = 1 \]

\[ x = 2 \]

\[ \neg \forall F p_{x=2} \]
Interval Values


Interval Temporal Logics

Let $\mathcal{T} = \langle T, < \rangle$ be a frame and $I(\mathcal{T}) = \{[a, b] \mid a, b \in T$ and $a \leq b\}$ be the set of intervals over $T$

Models are $\mathcal{M} = \langle I(\mathcal{T}), \{R_X\}, V \rangle$ where $R_X \subseteq I(\mathcal{T}) \times I(\mathcal{T})$ and $V : \text{At} \rightarrow \wp(I(\mathcal{T}))$. 
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- $\mathcal{M}, [a, b] \models p$ iff $[a, b] \in V(p)$
- $\mathcal{M}, [a, b] \models pt$ iff $a = b$
- $\mathcal{M}, [a, b] \models \langle X \rangle \varphi$ iff there is an interval $[c, d]$ such that $[a, b] R_X [c, d]$ and $\mathcal{M}, [c, d] \models \varphi$
Intervals of the form associate a modal operator order, often called (excluding the equality) between two intervals in a linear either semantics. There are 12 different non-trivial relations interval semantics these are excluded, the resulting semantics is called the connectives, like the strict semantics is adopted. The other propositional with formulae built over a set unary operators associated with all Allen's relations. For results for fragments of variants will play a central role in our technical results; sub-interval of two variants, namely, sub-interval (Halpern and Shoham's logic

\[ \langle A \rangle \quad [a, b] R_A [c, d] \iff b = c \]
\[ \langle L \rangle \quad [a, b] R_L [c, d] \iff b < c \]
\[ \langle B \rangle \quad [a, b] R_B [c, d] \iff a = c, d < b \]
\[ \langle E \rangle \quad [a, b] R_E [c, d] \iff b = d, a < c \]
\[ \langle D \rangle \quad [a, b] R_D [c, d] \iff a < c, d < b \]
\[ \langle O \rangle \quad [a, b] R_O [c, d] \iff a < c < b < d \]
High Undecidability!

D. Bersolin et al.. *The dark side of interval temporal logic: sharpening the undecidability border*. 2011.
Actions and Abilities: Pre-theoretic Intuitions

What does it mean for an agent to be able to do some action $a$ or bring about some state of affairs $\varphi$?
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Abilities

$Abl_i \varphi$: agent $i$ has the ability to bring about (see to it that) $\varphi$ is true

What are core logical principles?
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6. $Abl_i Abl_j \varphi \rightarrow Abl_i \varphi$, $Abl_i Abl_i \varphi \rightarrow Abl_i \varphi$
Games: \((Abl_i \varphi \land Abl_i \psi) \not\rightarrow Abl_i (\varphi \land \psi)\)
Games: $(Abl_i\varphi \land Abl_i\psi) \not\rightarrow Abl_i(\varphi \land \psi)$
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Games: \((Abl_i \varphi \land Abl_i \psi) \not\rightarrow Abl_i (\varphi \land \psi)\)

\[
\begin{array}{c}
A \\
\text{(} s_1 \text{)} \\
\text{(} p, q \text{)} \\
B \\
B \\
\text{(} s_2 \text{)} \\
\text{(} p \text{)} \\
B \\
\text{(} s_3 \text{)} \\
\text{(} q \text{)} \\
\end{array}
\]

\[s_1 \models Abl_A p \land Abl_A q\]
Games: \((Abl_i \varphi \land Abl_i \psi) \not\Rightarrow Abl_i(\varphi \land \psi)\)

\[
\begin{array}{c}
A \\
\downarrow \quad \downarrow \\
B & B \\
\downarrow & \downarrow \\
p & p, q & p, q & q \\
\end{array}
\]

\[s_1 \models Abl_A p \land Abl_A q \land \neg Abl_A (p \land q)\]
Games: \((\text{Abl}_i \varphi \land \text{Abl}_i \psi) \not\rightarrow \text{Abl}_i (\varphi \land \psi)\)


Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull’s eye.
Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull’s eye.

Intuitively, we do not want to say that Ann has the ability to throw a bull’s eye even though it happens to be true.
Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.
Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.
Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.
Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, it seems true that Ann does not have the ability to hit the top half of the dart board, and also she does not have the ability to hit the bottom half of the dart board.
Abilities

*Abl*$_i$*φ*: agent *i* has the ability to bring about (see to it that) *φ* is true

What are core logical principles? Depends very much on the intended “application” and how actions are represented...

1. *Abl*$_i$*φ* → *φ* (or *φ* → *Abl*$_i$*φ*)

2. ¬*Abl*$_i$*⊤*

3. (*Abl*$_i$*φ* ∧ *Abl*$_i$*ψ*) → *Abl*$_i$(*φ* ∧ *ψ*)

4. *Abl*$_i$(*φ* ∨ *ψ*) → (*Abl*$_i$*φ* ∨ *Abl*$_i$*ψ*)

5. *Abl*$_i$(*φ* ∧ *ψ*) → (*Abl*$_i$*φ* ∧ *Abl*$_i$*ψ*)

6. *Abl*$_i$*Abl*$_j$*φ* → *Abl*$_i$*φ*, *Abl*$_i$*Abl*$_i$*φ* → *Abl*$_i$*φ*

A Minimal Logic of Abilities

$C\varphi$ means “the agent is capable of realizing $\varphi$”

$E\varphi$ means “the agent does bring about $\varphi$”
A Minimal Logic of Abilities

$C \varphi$ means “the agent is capable of realizing $\varphi$”

$E \varphi$ means “the agent does bring about $\varphi$”

1. All propositional tautologies
2. $\neg C \top$
3. $E \varphi \land E \psi \rightarrow E (\varphi \land \psi)$
4. $E \varphi \rightarrow \varphi$
5. $E \varphi \rightarrow C \varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E \varphi \leftrightarrow E \psi$ and from $\varphi \leftrightarrow \psi$ infer $C \varphi \leftrightarrow C \psi$
Brief digression: weak systems of modal logic
PC Propositional Calculus

E $\square \varphi \leftrightarrow \neg \Diamond \neg \varphi$

M $\square (\varphi \land \psi) \rightarrow (\square \varphi \land \square \psi)$

C $(\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi)$

N $\square \top$

K $\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$

RE \[\begin{array}{c} \varphi \leftrightarrow \psi \\ \hline \square \varphi \leftrightarrow \square \psi \end{array}\]

Nec \[\begin{array}{c} \varphi \\ \hline \square \varphi \end{array}\]

MP \[\begin{array}{c} \varphi \varphi \rightarrow \psi \\ \hline \psi \end{array}\]
A modal logic $L$ is classical if it contains all instances of $E$ and is closed under $RE$. 

PC Propositional Calculus

$E \quad \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi$

$M \quad \Box(\varphi \land \psi) \rightarrow (\Box \varphi \land \Box \psi)$

$C \quad (\Box \varphi \land \Box \psi) \rightarrow \Box(\varphi \land \psi)$

$N \quad \Box \top$

$K \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$

$RE \quad \frac{\varphi \leftrightarrow \psi}{\Box \varphi \leftrightarrow \Box \psi}$

$Nec \quad \frac{\varphi}{\Box \varphi}$

$MP \quad \frac{\varphi \varphi \rightarrow \psi}{\psi}$
A modal logic $L$ is classical if it contains all instances of $E$ and is closed under $RE$.

$E$ is the smallest classical modal logic.
**PC** Propositional Calculus

\[ E \quad \square \phi \leftrightarrow \neg \Diamond \neg \phi \]

\[ M \quad \square (\phi \land \psi) \rightarrow (\square \phi \land \square \psi) \]

\[ C \quad (\square \phi \land \square \psi) \rightarrow \square (\phi \land \psi) \]

\[ N \quad \square \top \]

\[ K \quad \square (\phi \rightarrow \psi) \rightarrow (\square \phi \rightarrow \square \psi) \]

\[ \textbf{RE} \quad \phi \leftrightarrow \psi \]

\[ \quad \square \phi \leftrightarrow \square \psi \]

\[ \textbf{Nec} \quad \phi \]

\[ \quad \square \phi \]

\[ \textbf{MP} \quad \phi \quad \phi \rightarrow \psi \]

\[ \psi \]

**E** is the smallest *classical* modal logic.

In **E**, **M** is equivalent to

\[(\text{Mon}) \quad \frac{\phi \rightarrow \psi}{\square \phi \rightarrow \square \psi} \]
**PC** Propositional Calculus

**E** \( \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \)

**Mon**

\[
\frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}
\]

\( C \ (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \)

**N** \( \square \top \)

**K** \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)

**RE**

\[
\frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi}
\]

**Nec** \( \frac{\varphi}{\square \varphi} \)

**MP**

\[
\frac{\varphi \varphi \rightarrow \psi}{\psi}
\]

**E** is the smallest **classical** modal logic.

**EM** is the logic \( \text{E} + \text{Mon} \)
**PC**  6. Propositional Calculus

\[ E \quad \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi \]

**Mon** \[
\frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi}
\]

**C** \[
(\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)
\]

**N** \[
\Box T
\]

**K** \[
\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)
\]

**RE** \[
\frac{\varphi \leftrightarrow \psi}{\Box \varphi \leftrightarrow \Box \psi}
\]

**Nec** \[
\frac{\varphi}{\Box \varphi}
\]

**MP** \[
\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}
\]

\[ E \] is the smallest classical modal logic.

\[ EM \] is the logic \( E + Mon \)

\[ EC \] is the logic \( E + C \)
**PC** Propositional Calculus

**E** $\square \varphi \leftrightarrow \neg \lozenge \neg \varphi$

**Mon**

$\varphi \rightarrow \psi$

$\square \varphi \rightarrow \square \psi$

$\varphi \rightarrow \psi$

$\square \varphi \rightarrow \square \psi$

PC is the smallest **classical** modal logic.

**EM** is the logic $\text{E} + \text{Mon}$

**EC** is the logic $\text{E} + \text{C}$

**EMC** is the smallest **regular** modal logic

**RE**

$\varphi \leftrightarrow \psi$

$\square \varphi \leftrightarrow \square \psi$

**Nec**

$\varphi$

$\square \varphi$

**MP**

$\varphi \varphi \rightarrow \psi$

$\psi$

$\text{E}$ is the smallest classical modal logic.
**PC** Propositional Calculus

**E** \( \square \varphi \leftrightarrow \neg \lozenge \neg \varphi \)

**Mon**

\[
\frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}
\]

**C** \( (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \)

**N** \( \square \top \)

**K** \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)

**RE**

\[
\frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi}
\]

**Nec**

\[
\frac{\varphi}{\square \varphi}
\]

**MP**

\[
\frac{\varphi \varphi \rightarrow \psi}{\psi}
\]

**E** is the smallest classical modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest regular modal logic.

A logic is **normal** if it contains all instances of **E**, **C** and is closed under **Mon** and **Nec**
**PC**  Propositional Calculus

**E**  \[ \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \]

**Mon**  \[
\begin{array}{c}
\varphi \rightarrow \psi \\
\hline
\square \varphi \rightarrow \square \psi
\end{array}
\]

**C**  \[
(\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi)
\]

**N**  \[ \square \top \]

**K**  \[
\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)
\]

**RE**  \[
\begin{array}{c}
\varphi \leftrightarrow \psi \\
\hline
\square \varphi \leftrightarrow \square \psi
\end{array}
\]

**Nec**  \[
\begin{array}{c}
\varphi \\
\hline
\square \varphi
\end{array}
\]

**MP**  \[
\begin{array}{c}
\varphi \\
\varphi \rightarrow \psi \\
\hline
\psi
\end{array}
\]

**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest regular modal logic

**K** is the smallest normal modal logic
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC</td>
<td>Propositional Calculus</td>
</tr>
<tr>
<td>E</td>
<td>$\square \varphi \leftrightarrow \neg \Diamond \neg \varphi$</td>
</tr>
<tr>
<td>Mon</td>
<td>$\varphi \rightarrow \psi$ \hspace{1cm} $\square \varphi \rightarrow \Box \psi$</td>
</tr>
<tr>
<td>C</td>
<td>$(\square \varphi \land \square \psi) \rightarrow \Box (\varphi \land \psi)$</td>
</tr>
<tr>
<td>N</td>
<td>$\Box \top$</td>
</tr>
<tr>
<td>K</td>
<td>$\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$</td>
</tr>
<tr>
<td>RE</td>
<td>$\varphi \leftrightarrow \psi$ \hspace{1cm} $\Box \varphi \leftrightarrow \Box \psi$</td>
</tr>
<tr>
<td>Nec</td>
<td>$\varphi$ \hspace{1cm} $\Box \varphi$</td>
</tr>
<tr>
<td>MP</td>
<td>$\varphi \varphi \rightarrow \psi$ \hspace{1cm} $\psi$</td>
</tr>
</tbody>
</table>

**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest **regular** modal logic

**K = EMCN**
**PC**  Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \]

\[ Mon \quad \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi} \]

\[ C \quad (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \]

\[ N \quad \square T \]

\[ K \quad \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

\[ RE \quad \frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi} \]

\[ Nec \quad \frac{\varphi}{\square \varphi} \]

\[ MP \quad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \]

**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest **regular** modal logic

\[ K = PC(+E) + K + \text{Nec} + MP \]