Problem Set # 2

1. In this problem we consider a possible definition of common belief, analogous to the definition of common knowledge. Suppose there are two agents and a belief model \( \langle W, \{R_1, R_2\}, V \rangle \) where \( R_1 \) and \( R_2 \) are serial, transitive and Euclidean relations. Let \( R_B = (R_1 \cup R_2)^+ \), where \( R^+ \) is the transitive closure of \( R \) (the smallest transitive relation containing \( R \)). Define the common belief operator \( C^B \) as follows:

\[
\mathcal{M}, w \models C^B \varphi \iff \text{for each } v \in W, \text{ if } wR_B v \text{ then } \mathcal{M}, v \models \varphi
\]

(a) Provide a KD45 model \( \mathcal{M} = \langle W, \{R_1, R_2\}, V \rangle \) and a state \( w \in W \) where \( \mathcal{M}, w \models B_1(C^B p) \) but \( \mathcal{M}, w \models \neg C^B p \) (i.e., a state where agent 1 believes that \( p \) is commonly believed, but \( p \) is, in fact, not commonly believed).

(b) Provide an example that shows that negative introspection for common belief (\( \neg C^B \varphi \rightarrow C^B \neg C^B \varphi \)) is not valid.

2. We have argued that \( K_i \varphi \rightarrow K_j \varphi \) is valid on a frame \( \langle W, \{R_i\}_{i \in A} \rangle \) iff for each \( i, j \in A, R_j \subseteq R_i \). Find a property on frames \( \langle W, \{R_i\}_{i \in A} \rangle \) that guarantees that \( K_i \varphi \rightarrow K_i K_j \varphi \) is valid.

3. For a Bayesian model with a common prior \( \langle W, \{\sim_i\}_{i \in A}, \pi \rangle \), prove that for each \( i \in A \), \( \pi(E \mid B_i^p(E)) \geq p \).

4. Explain why Aumann’s original agreeing to disagree theorem (Theorem 7 in the handout for lecture 8) follows from Samet’s generalized agreeing to disagree theorem (Theorem 4 in the handout for lecture 8). Hint: fix and event \( E \subseteq W \) and for each agent \( i \), let the decision function \( d_i \) be defined as follows: \( d_i(w) = \pi(E \mid [w]_i) \) (the posterior probability of \( E \) for agent \( i \) at state \( w \)). Prove that \( d \) satisfies the ISTP.

The homework is DUE Wednesday, October 5.