1. **Linear time models**: A linear time model is a tuple $\mathcal{M} = \langle T, <, V \rangle$ where $T$ is a set of time points (or moments), $\langle \subseteq T \times T$ is the precedence relation: $s < t$ (“time point occurs earlier than $t$”) is irreflexive and transitive, and $V : \text{At} \to \wp(\mathcal{T})$ is a valuation function (describing when the atomic propositions are true). The linear time language is given by the following grammar:

$$p \mid \neg \varphi \mid \varphi \land \psi \mid G \varphi \mid H \varphi$$

where $p \in \text{At}$ (a countable set of atomic propositions). Truth is defined as follows:

- $\mathcal{M}, t \models p$ iff $t \in V(p)$
- $\mathcal{M}, t \models \neg \varphi$ iff $\mathcal{M}, t \not\models \varphi$
- $\mathcal{M}, t \models \varphi \land \psi$ iff $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \psi$
- $\mathcal{M}, t \models G \varphi$ iff for all $s \in T$, if $t < s$ then $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models H \varphi$ iff for all $s \in T$, if $s < t$ then $\mathcal{M}, s \models \varphi$

We define $F \varphi := \neg G \neg \varphi$ and $P \varphi := \neg H \neg \varphi$, so truth for these operators is:

- $\mathcal{M}, t \models F \varphi$ iff there is a $s \in T$ such that $t < s$ and $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models P \varphi$ iff there is a $s \in T$ such that $s < t$ and $\mathcal{M}, s \models \varphi$

We say $\varphi$ is valid on a temporal model $\mathcal{M} = \langle T, <, V \rangle$ provided $\mathcal{M}, t \models \varphi$ for all $t \in T$, and $\varphi$ is valid on a temporal frame $\langle T, < \rangle$, provided $\varphi$ is valid on every model based on $\langle T, < \rangle$ (these are standard definitions — see the notes on modal logic).

(a) A temporal frame $\langle T, < \rangle$ is **past-linear** provided for all $x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$.

**Claim 1** $FP\varphi \rightarrow (F \varphi \lor \varphi \lor P \varphi)$ is valid on $\langle T, < \rangle$ iff $\langle T, < \rangle$ is past-linear.

**Proof.** Suppose that $\mathcal{T} = \langle T, < \rangle$ is past-linear and $\mathcal{M} = \langle T, <, V \rangle$ is a model based on $\mathcal{T}$. We must show $FP\varphi \rightarrow (F \varphi \lor \varphi \lor P \varphi)$ is valid on $\mathcal{M}$. Let $t \in T$ be any moment and suppose that $\mathcal{M}, t \models FP\varphi$. Then, there is a $s \in T$ such that $t < s$ and $\mathcal{M}, s \models P \varphi$. This implies there is a $s'$ such that $s' < s$ with $\mathcal{M}, s' \models \varphi$. Since $\mathcal{T}$ is past-linear and $t < s$ and $s' < s$ we have three cases: either $t < s'$ or $t = s'$ or $s' < t$. In the first case $\mathcal{M}, t \models F \varphi$, in the second case $\mathcal{M}, t \models \varphi$ and in the third case $\mathcal{M}, t \models P \varphi$. Hence, $\mathcal{M}, t \models F \varphi \lor \varphi \lor P \varphi$, as desired.

Suppose that $\mathcal{T} = \langle T, < \rangle$ is not past-linear. Then, there are moments $s, s'$, and $t$ such that $s < t$, $s' < t$ but $s \neq s'$, $s \neq s'$ and $s' \neq s$. Let $\mathcal{M} = \langle T, <, V \rangle$ be a model based on $T$ where $V(p) = \{s'\}$. Since, $s' < t$ and $\mathcal{M}, s' \models p$, we have $\mathcal{M}, t \models Pp$. Then, since $s < t$, we have $\mathcal{M}, s \models FPp$. Note that $\mathcal{M}, s \models \neg Pp \land p \land \neg Fp$ (this follows since the only state satisfying $p$ is $s'$ and $s'$ is incomparable with $s$). Hence, $\mathcal{M}, s \not\models FPp \rightarrow (Pp \lor p \lor Fp)$.

QED
2. **Branching-time temporal models**: Given a temporal model \( \langle T, <, V \rangle \) a **branch** \( b \) is a maximal linearly ordered set of moments. We say \( s \in T \) is **on a branch** \( b \) of \( T \) provided \( s \in b \) (we also say “\( b \) is a branch going through \( t \)”). The branching time language is given by the following grammar:

\[
p | \neg \varphi | \varphi \land \psi | G \varphi | H \varphi | \Box \varphi
\]

where \( p \in \text{At} \). Truth is defined at pairs \( t/b \) where \( t \) is a moment on branch \( b \):

- \( \mathcal{M}, t/b \models p \) iff \( t/b \in V(p) \)
- \( \mathcal{M}, t/b \models \neg \varphi \) iff \( \mathcal{M}, t/b \not\models \varphi \)
- \( \mathcal{M}, t/b \models \varphi \land \psi \) iff \( \mathcal{M}, t/b \models \varphi \) and \( \mathcal{M}, t/b \models \psi \)
- \( \mathcal{M}, t/b \models G \varphi \) iff for all \( s \in T \), if \( s \) is on \( b \) and \( t < s \) then \( \mathcal{M}, s/b \models \varphi \)
- \( \mathcal{M}, t/b \models H \varphi \) iff for all \( s \in T \), if \( s \) is on \( b \) and \( s < t \) then \( \mathcal{M}, s/b \models \varphi \)
- \( \mathcal{M}, t/b \models \Box \varphi \) iff for all branches \( c \) through \( t \), \( \mathcal{M}, s/c \models \varphi \)

For each of the following formulas, determine which are valid on all temporal frames (for those that are not valid, provide counterexamples):

(a) \( \Diamond F \varphi \to F \Diamond \varphi \) is not valid.

**Proof.** Let \( T = \{t_1, t_2, t_3\} \) with \( t_1 < t_2 \) and \( t_1 < t_3 \), so there are two branches \( b = \{t_1, t_2\} \) and \( b' = \{t_1, t_3\} \). Let \( V(p) = \{t_2/b\} \). Then, \( \mathcal{M}, t_1/b \models Fp \) and so \( \mathcal{M}, t_1/b' \models \Diamond Fp \). However, since \( b' \) is the only branch going through \( t_3 \) and \( \mathcal{M}, t_3/b' \not\models p \), we have \( \mathcal{M}, t_3/b' \not\models \Diamond p \). Furthermore, since \( t_3 \) is the only moment on \( b' \) such that \( t_1 < t_3 \), we have \( \mathcal{M}, t_1/b' \not\models Fp \). Hence, \( \Diamond Fp \to F \Diamond p \) is not valid. This model is pictured below:

\[\text{QED}\]

(b) \( \Box F \varphi \to F \Box \varphi \) is not valid.

**Proof.** Suppose that \( T = \{t_1, t_2, t_3, t_4\} \) with \( t_1 < t_2 < t_3 \) and \( t_1 < t_2 < t_4 \). There are two branches: \( b_1 = \{t_1, t_2, t_3\} \) and \( b_2 = \{t_1, t_2, t_4\} \). Suppose that \( V(p) = \{t_2/b_1, t_4/b_2\} \). Then, since \( t_1 < t_2 \) and \( t_1 < t_4 \), we have \( \mathcal{M}, t_1/b_1 \models Fp \) and \( \mathcal{M}, t_1/b_2 \models Fp \). Hence, \( \mathcal{M}, t_1/b_1 \models \Box Fp \). However, since \( \mathcal{M}, t_3/b_1 \not\models \Box p \) (this follows from the fact that \( \mathcal{M}, t_3/b_1 \not\models p \) and \( b_1 \) is the only branch through \( t_3 \)) and \( \mathcal{M}, t_2/b_1 \not\models \Box p \) (this follows since \( \mathcal{M}, t_2/b_2 \not\models p \)), we have \( \mathcal{M}, t_1/b_1 \not\models F \Box p \). Therefore, \( \Box F \varphi \to F \Box \varphi \) is not valid. This model is pictured below:
(c) $F\Diamond \varphi \rightarrow \Diamond F\varphi$ is valid.

**Proof.** Suppose that $M, t/b \models F\Diamond \varphi$. Then there is a $t' \in b$ such that $t < t'$ and $M, t'/b \models \Diamond \varphi$. This implies there is a branch $c$ going through $t'$ such that $M, t'/c \models \varphi$. Since $t'$ is $t < t'$, any branching going through $t'$ must also go through $t$ (recall that branches are maximal sets of linearly ordered moments), so $c$ is a branching going through $t$. Since $M, t'/c \models \varphi$ and $t < t'$, we have $M, t/c \models F\varphi$. Since both $c$ and $b$ go through $t$, we have $M, t/b \models \Diamond F\varphi$. Hence, $F\Diamond \varphi \rightarrow \Diamond F\varphi$ is valid.

QED

(d) $F\Box \varphi \rightarrow \Box F\varphi$ is not valid.

**Proof.** Suppose that $T = \{t_1, t_2, t_3, t_4\}$ with $t_1 < t_2 < t_3$ and $t_1 < t_2 < t_4$. There are two branches: $b_1 = \{t_1, t_2, t_3\}$ and $b_2 = \{t_1, t_2, t_4\}$. Suppose that $V(p) = \{t_3/b_1\}$. Since $M, t_3/b_1 \models p$ and $b_1$ is the only branch through $t_3$, we have $M, t_3/b_1 \models \Box p$. Hence, $M, t_1/b_1 \models F\Box p$. However, since $M, t_4/b_2 \not\models p$ and $M, t_2/b_2 \not\models p$, we have $M, t_1/b_2 \not\models Fp$ and so $M, t_1/b_1 \not\models \Box Fp$. This model is pictured below:

QED

3. **Logics of Ability.** The logics of ability models of Brown are tuples $(W, R, V)$ where $R \subseteq W \times \wp(W)$ is a relation between states and subsets of $W$ (which Brown calls “clusters”) and $V : \text{At} \rightarrow \wp(W)$ a valuation function. The ability language is generated by the following grammar:

$$p \mid \neg \varphi \mid \varphi \land \psi \mid \langle \rangle \varphi \mid \langle \rangle \langle \rangle \varphi$$

where $p \in \text{At}$. The intended meaning is that $\langle \rangle \varphi$ expresses “the agent is able to bring about a state where $\varphi$ is true” and $\langle \rangle \langle \rangle \varphi$ is the weaker claim that “the agent is able to do something consistent with $\varphi$”. Truth is defined as follows:
• \( M, w \models p \) iff \( w \in V(p) \)
• \( M, w \models \neg \varphi \) iff \( M, w \not\models \varphi \)
• \( M, w \models \varphi \land \psi \) iff \( M, w \models \varphi \) and \( M, w \models \psi \)
• \( M, t \models \langle \rangle \varphi \) iff there is a \( X \subseteq W \) such that \( wRX \) and for all \( v \in X, M, v \models \varphi \)
• \( M, t \models \langle \rangle \varphi \) iff there is a \( X \subseteq W \) such that \( wRX \) and there is a \( v \in X \) such that \( M, v \models \varphi \)

Answer the following questions:

(a) Give a counter-model to \( \langle \rangle (\varphi \land \psi) \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \).
   **Answer.** Let \( W = \{w_1, w_2\} \) and suppose that \( V(p) = \{w_1\} \) and \( V(q) = \{w_3\} \). Let \( R \subseteq W \times \varphi(W) \) be such that \( w_1R\{w_1, w_2\} \). Then we have \( M, w_1 \models \langle \rangle (p \lor q) \) since \( w_1R\{w_1, w_2\} \subseteq [p \lor q]_M = [p]_M \cup [q]_M = \{w_1\} \cup \{w_2\} = \{w_1, w_2\} \). However, \( M, w_1 \not\models \langle \rangle p \) since \( \{w_1, w_2\} \not\subseteq [p]_M = \{w_1\} \), and similarly \( M, w_1 \not\models \langle \rangle q \). Hence, \( M, w_1 \not\models \langle \rangle (p \lor q) \rightarrow (\langle \rangle p \lor (\langle \rangle q)) \).

(b) Prove that \( \langle \rangle (\varphi \lor \psi) \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \) is valid.
   **Proof.** Suppose that \( M, w \models \langle \rangle (\varphi \lor \psi) \) then there is a \( X \subseteq W \) such that \( wRX \) and \( X \subseteq [\varphi \lor \psi]_M = [\varphi]_M \cup [\psi]_M \). Note that either \( X \cap [\varphi]_M \neq \emptyset \) or \( X \cap [\varphi]_M = \emptyset \). In the first case, \( M, w \models \langle \rangle \varphi \). In the second case, since \( X \cap [\varphi]_M = \emptyset \) and \( X \subseteq [\varphi]_M \cup [\psi]_M \), we have \( X \subseteq [\psi]_M \). Hence, \( M, w \models \langle \rangle \psi \). Thus, in either case, \( M, w \models \langle \rangle (\varphi \lor \psi) \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \). QED

(c) Is \( \langle \rangle \varphi \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \) valid? If it is, give a proof, and if it is not valid, give a property that would make it valid.
   **Answer.** No, \( \langle \rangle \varphi \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \) is not valid. Let \( M = \langle W, R, V \rangle \) be a model where there is a state \( w \) with \( wR\emptyset \). Then for any formula \( \varphi \), we have \( M, w \models \langle \rangle \varphi \), but \( M, w \not\models (\langle \rangle \varphi) \). It is not hard to see that if we assume that for all \( w \) we do not have \( wR\emptyset \), then \( \langle \rangle (\varphi \lor \psi) \rightarrow (\langle \rangle \varphi \lor (\langle \rangle \psi)) \) is valid.

4. **STIT models:** A stit model is a tuple \( M = \langle T, <, Choice, V \rangle \) where \( \langle T, <, V \rangle \) is a temporal model (defined as above), and \( Choice : A \times T \rightarrow \wp(H_t) \) is a function mapping each model to a partition of \( H_t \) (\( H_t \) is the set of branches going through \( t \)) satisfying the following conditions (we write \( Choice_t^i \) for \( Choice(i, t) \)):

- \( Choice_t^i \neq \emptyset \)
- \( K \neq \emptyset \) for each \( K \in Choice_t^i \)
- For all \( t \) and mappings \( s_t : A \rightarrow \varphi(H_t) \) such that \( s_t(i) \in Choice_t^i \), we have \( \bigcap_{s \in A} s_t(i) \neq \emptyset \)

The STIT language is defined according to the following grammar:
\[ \varphi = p \mid \neg \varphi \mid \varphi \land \psi \mid [i \text{ stit}]\varphi \mid \Box \varphi \]

where \( p \in \text{At} \). Truth is defined as follows:

- \( \mathcal{M}, t/h \models \varphi \) iff \( t/h \in V(p) \)
- \( \mathcal{M}, t/h \models \neg \varphi \) iff \( \mathcal{M}, t/h \not\models \varphi \)
- \( \mathcal{M}, t/h \models \varphi \land \psi \) iff \( \mathcal{M}, t/h \models \varphi \) and \( \mathcal{M}, t/h \models \psi \)
- \( \mathcal{M}, t/h \models \Box \varphi \) iff \( \mathcal{M}, t/h' \models \varphi \) for all \( h' \in H_t \)
- \( \mathcal{M}, t/h \models [i \text{ stit}]\varphi \) iff \( \mathcal{M}, t/h' \models \varphi \) for all \( h' \in \text{Choice}_i^t(h) \) (\( \text{Choice}_i^t(h) \) is the partition cell of \( \text{Choice}_i^t \) containing \( h \))

Define \( (i \text{ stit})\varphi \) to be \( \neg [i \text{ stit}]\neg \varphi \) and \( \Diamond \varphi \) to be \( \neg \Box \neg \varphi \). Answer the following two questions: Suppose that there are only two agents \( \mathcal{A} = \{1, 2\} \), then

(a) Prove that \( \Diamond \varphi \to (1 \text{ stit})(2 \text{ stit})\varphi \) is valid.

**Proof.** Suppose that \( \mathcal{M}, t/h \models \Diamond \varphi \) then there is a \( h' \in H_t \) such that \( \mathcal{M}, t/h' \models \varphi \). Consider the selection \( s_i(1) = \text{Choice}_i^t(h) \) (agent 1’s choice at \( h/t \)) and \( s_i(2) = \text{Choice}_i^t(h') \) (agent 2’s choice at \( t/h' \)). Then by the independence property, \( s_i(1) \cap s_i(2) \neq \emptyset \). So, there is a history \( h'' \in s_i(1) \cap s_i(2) = \text{Choice}_i^t(h) \cap \text{Choice}_i^t(h') \). Then, since \( h' \in \text{Choice}_i^t(h'') \) (recall, \( \text{Choice}_i^t \) is a partition) and \( \mathcal{M}, t/h' \models \varphi \), we have \( \mathcal{M}, t/h'' \models (2 \text{ stit})\varphi \). Since \( h'' \in \text{Choice}_i^t(h) \), we have \( \mathcal{M}, t/h \models (1 \text{ stit})(2 \text{ stit})\varphi \).

QED

(b) Conclude that \( \Box \varphi \) is definable as \([1 \text{ stit}][2 \text{ stit}]\varphi \) (argue that \( \Box \varphi \leftrightarrow [1 \text{ stit}][2 \text{ stit}]\varphi \) can be derived from the above axiom using the S5 axioms for \( \Box \) and \( [i \text{ stit}] \), and the axiom \( \Box \varphi \to [i \text{ stit}]\varphi \).

**Proof.** We derive \( \Box \varphi \leftrightarrow [1 \text{ stit}][2 \text{ stit}]\varphi \) using the STIT axioms:

<table>
<thead>
<tr>
<th>Prop: all instances of propositional tautologies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S5 for ( \Box )</strong></td>
</tr>
<tr>
<td>( K_\Box: \Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi) )</td>
</tr>
<tr>
<td>( T_\Box: \Box \varphi \to \varphi )</td>
</tr>
<tr>
<td>( 4_\Box: \Box \varphi \to \Box \Box \varphi )</td>
</tr>
<tr>
<td>( 5_\Box: \neg \Box \varphi \to \Box \neg \varphi )</td>
</tr>
<tr>
<td>( \text{Nec}_\Box: \text{ for } \varphi, \text{ infer } \Box \varphi )</td>
</tr>
<tr>
<td>( \Box \neg [i \text{ stit}]: \Box \varphi \to [i \text{ stit}]\varphi )</td>
</tr>
<tr>
<td><strong>S5 for ([i \text{ stit}])</strong></td>
</tr>
<tr>
<td>( K_{[i \text{ stit}]}: [i \text{ stit}] (\varphi \to \psi) \to ([i \text{ stit}]\varphi \to [i \text{ stit}]\psi) )</td>
</tr>
<tr>
<td>( T_{[i \text{ stit}]}: [i \text{ stit}] \varphi \to \varphi )</td>
</tr>
<tr>
<td>( 4_{[i \text{ stit}]}: [i \text{ stit}] \varphi \to [i \text{ stit}][i \text{ stit}]\varphi )</td>
</tr>
<tr>
<td>( 5_{[i \text{ stit}]}: \neg [i \text{ stit}] \varphi \to [i \text{ stit}] \neg [i \text{ stit}] \varphi )</td>
</tr>
<tr>
<td>( \text{Nec}_{[i \text{ stit}]}: \text{ for } \varphi, \text{ infer } [i \text{ stit}]\varphi )</td>
</tr>
<tr>
<td><strong>Ind: ( (\bigwedge_{i \in \mathcal{A}} \Diamond [i \text{ stit}]\varphi_i) \to \Diamond (\bigwedge_{i \in \mathcal{A}} [i \text{ stit}]\varphi_i) )</strong></td>
</tr>
</tbody>
</table>
We make use of the following rules of propositional logic:

Prop Reasoning: Trans
\[
\begin{align*}
A & \rightarrow B \\
B & \rightarrow C \\
\hline
A & \rightarrow C
\end{align*}
\]

Prop Reasoning: Equiv
\[
\begin{align*}
A & \leftrightarrow B \\
\varphi[C/A] & \leftrightarrow \varphi[C/B] \quad (\varphi[C/A] \text{ is } \varphi \text{ with all occurrences of } C \text{ replaced with } A)
\end{align*}
\]

Below is a derivation of \( \square \varphi \rightarrow [1 \text{ stit}] [2 \text{ stit}] \varphi \):

1. \( \square \varphi \rightarrow [2 \text{ stit}] \varphi \) \quad Axiom \( \square \rightarrow [2 \text{ stit}] \)
2. \( \square (\square \varphi \rightarrow [2 \text{ stit}] \varphi) \) \quad \text{Nec} \( \square \)
3. \( \square (\square \varphi \rightarrow [2 \text{ stit}] \varphi) \rightarrow (\square \square \varphi \rightarrow \square [2 \text{ stit}] \varphi) \) \quad \text{Axiom } \( \text{K} \)
4. \( \square \square \varphi \rightarrow \square [2 \text{ stit}] \varphi \) \quad \text{MP 2,3}
5. \( \square \varphi \rightarrow \square \square \varphi \) \quad \text{Axiom } 4 \square
6. \( \square \varphi \rightarrow \square [2 \text{ stit}] \varphi \) \quad \text{Prop Reasoning: Trans 4, 5}
7. \( \square [2 \text{ stit}] \varphi \rightarrow [1 \text{ stit}] [2 \text{ stit}] \varphi \) \quad \text{Axiom } \( \square \rightarrow [1 \text{ stit}] \)
8. \( \square \varphi \rightarrow [1 \text{ stit}] [2 \text{ stit}] \varphi \) \quad \text{Prop Reasoning: Trans 6, 7}

Below is a derivation of \( [1 \text{ stit}] [2 \text{ stit}] \varphi \rightarrow \square \varphi \):

1. \( \lozenge \neg \varphi \rightarrow (1 \text{ stit}) \langle 2 \text{ stit} \rangle \neg \varphi \) \quad \text{Axiom}
2. \( \neg (1 \text{ stit}) \langle 2 \text{ stit} \rangle \neg \varphi \rightarrow \neg \lozenge \neg \varphi \) \quad \text{Prop reasoning}
3. \( \neg \neg [1 \text{ stit}] \neg \neg [2 \text{ stit}] \neg \neg \varphi \rightarrow \neg \lozenge \neg \varphi \) \quad \text{[i stit]-dual}
4. \( [1 \text{ stit}] [2 \text{ stit}] \varphi \rightarrow \neg \lozenge \neg \varphi \) \quad \text{Prop reasoning } (\neg \neg \varphi \leftrightarrow \varphi)
5. \( [1 \text{ stit}] [2 \text{ stit}] \varphi \rightarrow \square \varphi \) \quad \square \text{-dual}

QED

The homework is DUE Tuesday, November 22 (put your answers in my mailbox).