1. Prove that the following axiom of ceteris paribus logic is valid (see slide 22 of lecture 21 on 11/16):

\[(\alpha \land (\Gamma)^{\leq}(\alpha \land \varphi)) \rightarrow (\Gamma \cup \{\alpha\})^{\leq}\varphi\]

**Proof.** Let \(\mathcal{M}\) be a preference model and \(w\) a state in \(\mathcal{M}\). Suppose that \(\mathcal{M}, w \models \alpha \land (\Gamma)^{\leq}(\alpha \land \varphi)\). Then, \(\mathcal{M}, w \models \alpha\) and there is a \(v\) such that \(w \equiv_{\Gamma} v\) and \(w \leq v\) and \(\mathcal{M}, v \models \alpha \land \varphi\). Now we have for all \(\varphi \in \Gamma\), \(\mathcal{M}, w \models \varphi\) iff \(\mathcal{M}, v \models \varphi\) and \(\mathcal{M}, w \models \alpha\) and \(\mathcal{M}, v \models \alpha\). Hence, \(w \equiv_{\Gamma \cup \{\alpha\}} v\). Since \(w \leq v\), we have \(\mathcal{M}, w \models (\Gamma \cup \{\alpha\})^{\leq}\varphi\). Since \(\mathcal{M}\) and \(w\) were arbitrary, \((\alpha \land (\Gamma)^{\leq}(\alpha \land \varphi)) \rightarrow (\Gamma \cup \{\alpha\})^{\leq}\varphi\) is valid. \(\text{QED}\)

2. Let \(X, Y\) be subsets of \(W\) and suppose that \(\leq\) is a reflexive, connected and transitive order over \(W\). Say \(X \leq_{\forall\forall} Y\) provided for all \(x \in X\) and for all \(y \in Y\), we have \(x \leq y\). Assume that \(\leq\) is reflexive, transitive and complete, is \(\leq_{\forall\forall}\) also reflexive, transitive, and complete? If so, prove it and if not, give a counterexample.

**Proof.** Suppose that \(\leq \subseteq W \times W\) is reflexive, transitive and connected. We show that \(\leq_{\forall\forall}\) is transitive but not reflexive nor connected.

\(\leq_{\forall\forall}\) **is not reflexive:** Suppose that \(W = \{1, 2, 3, 4\}\) with \(1 < 2 < 3 < 4\) (where \(i < j\) means \(i \leq j\) but \(j \not\leq i\)). Consider \(X = \{1, 2\}\), then \(X \not\leq_{\forall\forall} X\) since \(2 \not\leq 1\).

\(\leq_{\forall\forall}\) **is transitive** for all nonempty sets: First of all, not that for any sets \(X\) and \(Y\), \(X \leq_{\forall\forall} \emptyset\) and \(\emptyset \leq_{\forall\forall} Y\). Transitivity would imply \(X \leq_{\forall\forall} Y\), but it is easy to find counterexamples to this. Suppose that \(X, Y,\) and \(Z\) are nonempty. Suppose that \(X \leq_{\forall\forall} Y\) and \(Y \leq_{\forall\forall} Z\), we must show that \(X \leq_{\forall\forall} Z\). Let \(x \in X\) and \(z \in Z\). Since \(Y\) is nonempty there is an element \(y \in Y\). Since \(X \leq_{\forall\forall} Y\), we have \(x \leq y\). Since, \(Y \leq_{\forall\forall} Z\), we have \(y \leq z\). Since \(\leq\) is transitive, we have \(x \leq z\). Since \(x\) and \(z\) were arbitrary elements of \(X\) and \(Z\), respectively, we have \(X \leq_{\forall\forall} Z\).

\(\leq_{\forall\forall}\) **is not connected:** Suppose that \(W = \{1, 2, 3, 4\}\) with \(1 < 2 < 3 < 4\) (where \(i < j\) means \(i \leq j\) but \(j \not\leq i\)). Let \(X = \{1, 3\}\) and \(Y = \{2, 4\}\) then \(X \not\leq_{\forall} Y\) and \(Y \not\leq_{\forall} X\).

\(\text{QED}\)

Can you think of any other interesting principles that \(\leq_{\forall\forall}\) satisfies? One interesting set of principles are downward and upwards monotonicity:

- If \(X \leq_{\forall\forall} Y\) and \(Z \subseteq X\), then \(Z \leq_{\forall\forall} Y\).
- If \(X \leq_{\forall\forall} Y\) and \(Z \subseteq X\), then \(Z \leq_{\forall\forall} Z\).
3. Recall the model of knowledge and preference from Lecture 22 (on 11/21): \( \mathcal{M} = \langle W, \sim, \preceq, V \rangle \) where \( \sim \) is an equivalence relation and \( \preceq \) is a reflexive, transitive and total preference relation. Truth is defined as follows:

- \( \mathcal{M}, w \models K \varphi \) iff for all \( v \in W \), if \( w \sim v \) then \( \mathcal{M}, v \models \varphi \)
- \( \mathcal{M}, w \models \langle \preceq \rangle \varphi \) iff there is a \( v \in W \) with \( w \preceq v \) and \( \mathcal{M}, v \models \varphi \)
- \( \mathcal{M}, w \models A \varphi \) iff for all \( v \in W \), \( \mathcal{M}, v \models \varphi \)
- \( \mathcal{M}, w \models \langle \sim \cap \preceq \rangle \varphi \) iff there is a \( v \in W \) such that \( w \sim v \) and \( w \preceq v \) with \( \mathcal{M}, v \models \varphi \)

Given an example to show that \( K(\psi \rightarrow \langle \preceq \rangle \varphi) \) and \( K(\psi \rightarrow \langle \sim \cap \preceq \rangle \varphi) \) or not equivalent (i.e., find a model and state where one of the formulas is true, but the other is not true). It is easy to see that \( A(\psi \rightarrow \langle \preceq \rangle \varphi) \rightarrow K(\psi \rightarrow \langle \preceq \rangle \varphi) \) is valid (this is an instance of the validity \( A \varphi \rightarrow K \varphi \)), but what is the relationship between \( A(\psi \rightarrow \langle \preceq \rangle \varphi) \) and \( K(\psi \rightarrow \langle \sim \cap \preceq \rangle \varphi) \) (does one imply the other or are the two formulas independent)?

**Answer.** We can construct a model where \( K(p \rightarrow \langle \preceq \rangle q) \) is true but \( K(p \rightarrow \langle \sim \cap \preceq \rangle q) \) is false. The model is drawn below (with the undirected lines denoting the information relation \( \sim \) and the arrows denoting the preference relation where an arrow from \( w \) to \( v \) means \( w \preceq v \)).

![Model Diagram](attachment:image.png)

Then, \( \mathcal{M}, w \models K(p \rightarrow \langle \preceq \rangle q) \), but \( \mathcal{M}, w \not\models K(p \rightarrow \langle \sim \cap \preceq \rangle q) \)

**Claim 1** \( K(\psi \rightarrow \langle \sim \cap \preceq \rangle \varphi) \rightarrow K(\psi \rightarrow \langle \preceq \rangle \varphi) \) is valid

**Proof.** Suppose that \( \mathcal{M}, w \models K(\psi \rightarrow \langle \sim \cap \preceq \rangle \varphi) \). Suppose that there is a \( v \) such that \( w \sim v \). We must show \( \mathcal{M}, v \models \psi \rightarrow \langle \preceq \rangle \varphi \). Suppose that \( \mathcal{M}, v \models \psi \). Since, \( \mathcal{M}, w \models K(\psi \rightarrow \langle \sim \cap \preceq \rangle \varphi) \) and \( w \sim v \), we have \( \mathcal{M}, v \models \psi \rightarrow \langle \sim \cap \preceq \rangle \varphi \). This implies \( \mathcal{M}, v \models \langle \sim \cap \preceq \rangle \varphi \). Hence, there is a \( v' \) such that \( v(\sim \cap \preceq) v' \) and \( \mathcal{M}, v' \models \varphi \). Since, \( \langle \sim \cap \preceq \rangle \subseteq \preceq \), we have \( v \preceq v' \). Hence, \( \mathcal{M}, v \models \langle \preceq \rangle \varphi \), as desired. Hence, \( \mathcal{M}, w \models K(\psi \rightarrow \langle \preceq \rangle \varphi) \).
**Claim 2** \( A(\psi \to (\preceq)\varphi) \) and \( K(\psi \to (\sim \cap \preceq)\varphi) \) are independent

**Proof.** We can construct two models: one where \( A(p \to (\preceq)q) \) is true but \( K(p \to (\preceq \cap \sim)q) \) is false, and vice versa. The models are drawn below (with the undirected lines denoting the information relation \( \sim \) and the arrows denoting the preference relation where an arrow from \( w \) to \( v \) means \( w \preceq v \)).

**Example 1**

Then we have \( \mathcal{M}, w \models A(p \to (\preceq)q) \) but \( \mathcal{M}, w \not\models K(p \to (\sim \cap \preceq)q) \)

**Example 2**

Then we have \( \mathcal{M}, w \not\models A(p \to (\preceq)q) \) but \( \mathcal{M}, w \models K(p \to (\sim \cap \preceq)q) \)

\text{QED}