Subgames

Let $H = \langle H_1, \ldots, H_n, u_1, \ldots, u_n \rangle$ be an arbitrary strategic game.
Subgames

Let \( H = \langle H_1, \ldots, H_n, u_1, \ldots, u_n \rangle \) be an arbitrary strategic game.

A restriction of \( H \) is a sequence \( G = (G_1, \ldots, G_n) \) such that \( G_i \subseteq H_i \) for all \( i \in \{1, \ldots, n\} \).

The set of all restrictions of a game \( H \) ordered by componentwise set inclusion forms a complete lattice.
Game Models

Relational models: $\langle W, R_i \rangle$ where $R_i \subseteq W \times W$. Write $R_i(w) = \{ v \mid wR_i v \}$.

Events: $E \subseteq W$

Knowledge/Belief: $\Box E = \{ w \mid R_i(w) \subseteq E \}$

Common knowledge/belief:

$\Box^1 E = \Box E$
$\Box^{k+1} E = \Box \Box^k E$
$\Box* E = \bigcap_{k=1}^{\infty} \Box^k E$

Fact. An event $F$ is called evident provided $F \subseteq \Box F$. $w \in \Box* E$ provided there is an evident event $F$ such that $w \in F \subseteq \Box E$. 
Let $G = (G_1, \ldots, G_n)$ be a restriction of a game $H$.

A **knowledge/belief model of** $G$ is a tuple

$\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ where $\langle W, R_1, \ldots, R_n \rangle$ is a knowledge/belief model and $\sigma_i : W \rightarrow G_i$. 
Game Models

Let $G = (G_1, \ldots, G_n)$ be a restriction of a game $H$.

A knowledge/belief model of $G$ is a tuple $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ where $\langle W, R_1, \ldots, R_n \rangle$ is a knowledge/belief model and $\sigma_i : W \to G_i$.

Given a model $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ for a restriction $G$ and a sequence $\overline{E} = \{E_1, \ldots, E_n\}$ where $E_i \subseteq W$, $G_{\overline{E}} = (\sigma_1(E_1), \ldots, \sigma_n(E_n))$
Some Lattice Theory

- \((D, \subseteq)\) is a lattice with largest element \(\top\). \(T : D \to D\) an operator.

- \(T\) is monotonic if for all \(G, G'\), \(G \subseteq G'\) implies \(T(G) \subseteq T(G')\).

- \(G\) is a fixed-point if \(T(G) = G\).

- \(\nu_T\) is the largest fixed point of \(T\).

- \(T_\infty\) is the "outcome of \(T\)":
  \[
  T_0 = \top, \quad T_{\alpha + 1} = T(T_\alpha), \quad T_\beta = \bigcap_{\alpha < \beta} T_\alpha.
  \]
  The outcome of iterating \(T\) is the least \(\alpha\) such that \(T_\alpha + 1 = T_\alpha\), denoted \(T_\infty\).

- Tarski's Fixed-Point Theorem: Every monotonic operator \(T\) has a (least and largest) fixed point \(T_\infty = \nu_T = \bigcup \{G | G \subseteq T(G)\}\).

- \(T\) is contracting if \(T(G) \subseteq G\). Every contracting operator has an outcome (\(T_\infty\) is well-defined).
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Some Lattice Theory

- $(D, \subseteq)$ is a lattice with largest element $\top$. $T : D \to D$ an operator.
- $T$ is monotonic if for all $G, G'$, $G \subseteq G'$ implies $T(G) \subseteq T(G')$.
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- $T^\infty$ is the “outcome of $T$: $T^0 = \top$, $T^{\alpha+1} = T(T^\alpha)$, $T^\beta = \bigcap_{\alpha < \beta} T^\alpha$.
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- **Tarski’s Fixed-Point Theorem**: Every monotonic operator \(T\) has a (least and largest) fixed point \(T^\infty = \nu T = \bigcup\{G \mid G \subseteq T(G)\}\).
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- $(D, \subseteq)$ is a lattice with largest element $\top$. $T : D \to D$ an operator.
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- **Tarski’s Fixed-Point Theorem**: Every monotonic operator $T$ has a (least and largest) fixed point $T^\infty = \nu T = \bigcup \{G \mid G \subseteq T(G)\}$.
- $T$ is contracting if $T(G) \subseteq G$. Every contracting operator has an outcome ($T^\infty$ is well-defined).
Rationality Properties

\( \varphi(s_i, G_i, G_{-i}) \) holds between a strategy \( s_i \in H_i \), a set of strategies \( G_i \) for player \( i \) and strategies \( G_{-i} \) of the opponents. Intuitively \( s_i \) is \( \varphi \)-optimal strategy for player \( i \) in the restricted game \( \langle G_i, G_{-i}, u_1, \ldots, u_n \rangle \) (where the payoffs are suitably restricted).
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\( \varphi_i \) is **monotonic** if for all \( G_{-i}, G'_{-i} \subseteq H_{-i} \) and \( s_i \in H_i \)

\[ G_{-i} \subseteq G'_{-i} \text{ and } \varphi(s_i, H_i, G_{-i}) \text{ implies } \varphi(s_i, H_i, G'_{-i}) \]
If $\varphi = (\varphi_1, \ldots, \varphi_n)$, then define $T_\varphi(G) = G'$ where

- $G = (G_1, \ldots, G_n)$, $G' = (G'_1, \ldots, G'_n)$,
- for all $i \in \{1, \ldots, n\}$, $G'_i = \{s_i \in G_i \mid \varphi_i(s_i, H_i, G_{-i})\}$
Removing Strategies

If \( \varphi = (\varphi_1, \ldots, \varphi_n) \), then define \( T_\varphi(G) = G' \) where

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\( T_\varphi \) is contracting, so it has an outcome \( T_\varphi^\infty \)
Removing Strategies

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$T_\varphi$ is contracting, so it has an outcome $T_\varphi^\infty$

If each $\varphi_i$ is monotonic, then $\nu T_\varphi$ exists and equals $T_\varphi^\infty$. 
Rational Play

Let $H = \langle H_1, \ldots, H_n, u_1, \ldots, u_n \rangle$ a strategic game and $\langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle$ a model for $H$.

$\sigma_i(w)$ is the strategy player is using in state $w$.

$G_{R_i(w)}$ is a restriction of $H$ giving $i$’s view of the game.
Rational Play

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Player $i$ is $\varphi_i$-rational in the state $w$ if $\varphi_i(\sigma_i(w), H_i, (G_{R_i(w)})_{-i})$ holds.
Rational Play

Let \( H = \langle H_1, \ldots, H_n, u_1, \ldots, u_n \rangle \) a strategic game and \( \langle W, R_1, \ldots, R_n, \sigma_1, \ldots, \sigma_n \rangle \) a model for \( H \).

\( \sigma_i(w) \) is the strategy player is using in state \( w \).

\( G_{R_i(w)} \) is a restriction of \( H \) giving \( i \)'s view of the game.

Player \( i \) is \( \varphi_i \)-rational in the state \( w \) if \( \varphi_i(\sigma_i(w), H_i, (G_{R_i(w)})_{-i}) \) holds.

\( \text{Rat}(\varphi) = \{ w \in W \mid \text{each player is } \varphi_i \text{-rational in } w \} \)

\( \Box \text{Rat}(\varphi) \)
\( \Box^* \text{Rat}(\varphi) \)
Theorem (Apt and Zvesper).

- Suppose that each $\varphi_i$ is monotonic. Then for all belief models for $H$,
  \[ G_{\text{Rat}(\varphi) \cap B^*(\text{Rat}(\varphi))} \subseteq T^\infty \]

- Suppose that each $\varphi_i$ is monotonic. Then for all knowledge models for $H$,
  \[ G_{K^*(\text{Rat}(\varphi))} \subseteq T^\infty \]

- For some standard knowledge model for $H$,
  \[ T^\infty \subseteq G_{K^*(\text{Rat}(\varphi))} \]

Claim If each $\varphi_i$ is monotonic, then $G_{\text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)} \subseteq T^\infty$.
**Claim** If each $\varphi_i$ is monotonic, then $G_{\text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)} \subseteq T^\infty$.

Let $s_i$ be an element of the $i$th component of $G_{\text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)}$:

$s_i = \sigma_i(w)$ for some $w \in \text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)$.
**Claim** If each \( \varphi_i \) is monotonic, then \( G_{\text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)} \subseteq T^\infty \).

Let \( s_i \) be an element of the \( i \)th component of \( G_{\text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)} \):

\[
s_i = \sigma_i(w) \quad \text{for some } w \in \text{Rat}(\varphi) \cap \Box^* \text{Rat}(\varphi)
\]

there is an \( F \) such that \( F \subseteq \Box F \) and

\[
w \in F \subseteq \Box \text{Rat}(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi)\}
\]
Claim If each $\varphi_i$ is monotonic, then $G_{Rat(\varphi) \cap \Box^*Rat(\varphi)} \subseteq T^\infty$.

Let $s_i$ be an element of the $i$th component of $G_{Rat(\varphi) \cap \Box^*Rat(\varphi)}$:

$s_i = \sigma_i(w)$ for some $w \in Rat(\varphi) \cap \Box^*Rat(\varphi)$

there is an $F$ such that $F \subseteq \Box F$ and

$$w \in F \subseteq \Box Rat(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq Rat(\varphi)\}$$

Claim. $G_{F \cap Rat(\varphi)}$ is post-fixed point of $T_\varphi$

$(G_{F \cap Rat(\varphi)} \subseteq T_\varphi(G_{F \cap Rat(\varphi)}))$. 
**Claim** If each $\varphi_i$ is monotonic, then $G_{Rat(\varphi) \cap \square^*Rat(\varphi)} \subseteq T^\infty_{\varphi}$.

Let $s_i$ be an element of the $i$th component of $G_{Rat(\varphi) \cap \square^*Rat(\varphi)}$: $s_i = \sigma_i(w)$ for some $w \in Rat(\varphi) \cap \square^*Rat(\varphi)$

there is an $F$ such that $F \subseteq \square F$ and

$$w \in F \subseteq \square Rat(\varphi) = \{v \in W \mid \forall i R_i(v) \subseteq Rat(\varphi)\}$$

**Claim.** $G_{F \cap Rat(\varphi)}$ is post-fixed point of $T_\varphi$

($G_{F \cap Rat(\varphi)} \subseteq T_\varphi(G_{F \cap Rat(\varphi)})$).

Since each $\varphi_i$ is monotonic, $T_\varphi$ is monotonic and by Tarski’s fixed-point theorem, $G_{F \cap Rat(\varphi)} \subseteq T^\infty_\varphi$. But $s_i = \sigma_i(w)$ and $w \in F \cap Rat(\varphi)$, so $s_i$ is the $i$th component in $T^\infty_\varphi$. 
$F \subseteq \square F$ and $w \in F \subseteq \square \text{Rat}(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi)\}$

Claim. $G_{F \cap \text{Rat}(\varphi)}$ is post-fixed point of $T_\varphi$
$(G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}))$. 
\( F \subseteq \Box F \) and \( w \in F \subseteq \Box \text{Rat}(\varphi) = \{ v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi) \}\)

**Claim.** \( G_{F \cap \text{Rat}(\varphi)} \) is post-fixed point of \( T_\varphi \)
\( (G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}) \).

Let \( w' \in F \cap \text{Rat}(\varphi) \) and let \( i \in \{1, \ldots, n\} \).
$F \subseteq \square F$ and $w \in F \subseteq \square \text{Rat}(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi)\}$

**Claim.** $G_{F \cap \text{Rat}(\varphi)}$ is post-fixed point of $T_{\varphi}$

$(G_{F \cap \text{Rat}(\varphi)} \subseteq T_{\varphi}(G_{F \cap \text{Rat}(\varphi)})$).

Let $w' \in F \cap \text{Rat}(\varphi)$ and let $i \in \{1, \ldots, n\}$.

Since $w' \in \text{Rat}(\varphi)$, $\varphi_i(\sigma_i(w'), H_i, (G_{R_i(w)})_{-i})$ holds.
\[ F \subseteq \Box F \text{ and } w \in F \subseteq \Box \text{Rat}(\varphi) = \{ v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi) \} \]

Claim. \( G_{F \cap \text{Rat}(\varphi)} \) is post-fixed point of \( T_\varphi \)
\( (G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}) ) \).

Let \( w' \in F \cap \text{Rat}(\varphi) \) and let \( i \in \{1, \ldots, n\} \).

Since \( w' \in \text{Rat}(\varphi) \), \( \varphi_i(\sigma_i(w'), H_i, (G_{R_i(w)})^{-i}) \) holds.

\( F \) is evident, so \( R_i(w') \subseteq F \). We also have \( R_i(w') \subseteq \text{Rat}(\varphi) \).
\[ F \subseteq \Box F \text{ and } w \in F \subseteq \Box \text{Rat}(\varphi) = \{ v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi) \} \]

**Claim.** \( G_{F \cap \text{Rat}(\varphi)} \) is post-fixed point of \( T_\varphi \)

\( (G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)})) \).

Let \( w' \in F \cap \text{Rat}(\varphi) \) and let \( i \in \{1, \ldots, n\} \).

Since \( w' \in \text{Rat}(\varphi), \varphi_i(\sigma_i(w')), H_i, (G_{R_i(w)})_{-i} \) holds.

\( F \) is evident, so \( R_i(w') \subseteq F \). We also have \( R_i(w') \subseteq \text{Rat}(\varphi) \).

Hence, \( R_i(w') \subseteq F \cap \text{Rat}(\varphi) \).
\[ F \subseteq \Box F \text{ and } w \in F \subseteq \Box \text{Rat}(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi)\} \]

**Claim.** \( G_{F \cap \text{Rat}(\varphi)} \) is post-fixed point of \( T_\varphi \)
\( (G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}) \).

Let \( w' \in F \cap \text{Rat}(\varphi) \) and let \( i \in \{1, \ldots, n\} \).

Since \( w' \in \text{Rat}(\varphi) \), \( \varphi_i(\sigma_i(w'), H_i, (G_{R_i(w)})_{-i}) \) holds.

\( F \) is evident, so \( R_i(w') \subseteq F \). We also have \( R_i(w') \subseteq \text{Rat}(\varphi) \).

Hence, \( R_i(w') \subseteq F \cap \text{Rat}(\varphi) \).

This implies \( (G_{R_i(w')}) \subseteq (G_{F \cap \text{Rat}(\varphi)})_{-i} \), and so by monotonicity of \( \varphi_i, \varphi_i(s_i, H_i, (G_{F \cap \text{Rat}(\varphi)})_{-i}) \) holds.
\[ F \subseteq \Box F \text{ and } w \in F \subseteq \Box \text{Rat}(\varphi) = \{v \in W \mid \forall i \ R_i(v) \subseteq \text{Rat}(\varphi)\} \]

**Claim.** \( G_{F \cap \text{Rat}(\varphi)} \) is post-fixed point of \( T_\varphi \)
\[
(G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}))
\]

Let \( w' \in F \cap \text{Rat}(\varphi) \) and let \( i \in \{1, \ldots, n\} \).

Since \( w' \in \text{Rat}(\varphi), \varphi_i(\sigma_i(w')), H_i, (G_{R_i(w)})_{-i} \) holds.

\( F \) is evident, so \( R_i(w') \subseteq F \). We also have \( R_i(w') \subseteq \text{Rat}(\varphi) \).

Hence, \( R_i(w') \subseteq F \cap \text{Rat}(\varphi) \).

This implies \( (G_{R_i(w')}) \subseteq (G_{F \cap \text{Rat}(\varphi)})_{-i} \), and so by monotonicity of \( \varphi_i, \varphi_i(s_i, H_i, (G_{F \cap \text{Rat}(\varphi)})_{-i}) \) holds.

This means \( G_{F \cap \text{Rat}(\varphi)} \subseteq T_\varphi(G_{F \cap \text{Rat}(\varphi)}) \).
sd_i(s_i, G_i, G_{-i}) is \neg \exists s'_i \in G_i, \forall s_{-i} \in G_{-i} u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})
\[ sd_i(s_i, G_i, G_{-i}) \text{ is } \neg \exists s'_i \in G_i, \forall s_{-i} \in G_{-i} u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \]

\[ br_i(s_i, G_i, G_{-i}) \text{ is } \exists \mu_i \in B_i(G_{-i}) \forall s'_i \in G_i, U_i(s_i, \mu_i) \geq U_i(s'_i, \mu_i). \]
$sd_i(s_i, G_i, G_{-i})$ is $\neg \exists s_i' \in G_i, \forall s_{-i} \in G_{-i} u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$

$br_i(s_i, G_i, G_{-i})$ is $\exists \mu_i \in B_i(G_{-i}) \forall s_i' \in G_i, U_i(s_i, \mu_i) \geq U_i(s_i', \mu_i)$.

$U_{\varphi}(G) = G'$ where $G'_i = \{s_i \in G_i \mid \varphi_i(s_i, G_i, G_{-i})\}$. 

\[ sd_i(s_i, G_i, G_{-i}) \text{ is } \neg \exists s_i' \in G_i, \forall s_{-i} \in G_{-i} u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \]

\[ br_i(s_i, G_i, G_{-i}) \text{ is } \exists \mu_i \in B_i(G_{-i}) \forall s_i' \in G_i, U_i(s_i, \mu_i) \geq U_i(s_i', \mu_i). \]

\[ U_\varphi(G) = G' \text{ where } G'_i = \{ s_i \in G_i \mid \varphi_i(s_i, G_i, G_{-i}) \}. \]

Note: \( U_\varphi \) is \textit{not} monotonic.
**Corollary.** For all belief models, \( G_{\text{Rat}(br) \cap \Box^* \text{Rat}(br)} \subseteq U_{sd} \). For all \( G \), we have

\[
T_{br}(G) \subseteq T_{sd}(G)
\]

\[
T_{sd}(G) \subseteq U_{sd}(G)
\]

Then, \( T_{sd}^\infty \subseteq U_{sd}^\infty \).
**Corollary.** For all belief models, \( \mathcal{G}_{\mathbf{Rat}(br) \cap \square^* \mathbf{Rat}(br)} \subseteq U_{sd}^\infty \). For all \( G \), we have

\[
T_{br}(G) \subseteq T_{sd}(G)
\]

\[
T_{sd}(G) \subseteq U_{sd}(G)
\]

Then, \( T_{sd}^\infty \subseteq U_{sd}^\infty \).

**Fact.** Consider two operators \( T_1, T_2 \) on \( (D, \subseteq) \) such that,

- for all \( G \), \( T_1(G) \subseteq T_2(G) \)
- \( T_1 \) is monotonic
- \( T_2 \) is contracting

Then, \( T_1^\infty \subseteq T_2^\infty \).
This analysis does not work for weak dominance...