Vector Spaces and Linear Maps

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1 About

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2 Vector spaces

Vector spaces are the basic setting in which linear algebra happens. A vector space over a field $\mathbb{F}$ consists of a set V (the elements of which are called **vectors**) along with an addition operation $+ : V \times V \to V$ and a **scalar multiplication** operation $\mathbb{F} \times V \to V$ satisfying

(i) There exists an additive identity, denoted 0, in V such that $v + 0 = v$ for all $v \in V$

(ii) For each $v \in V$, there exists an additive inverse, denoted $-v$, such that $v + (-v) = 0$

(iii) There exists a multiplicative identity, denoted 1, in $\mathbb{F}$ such that $1v = v$ for all $v \in V$

(iv) Commutativity: $u + v = v + u$ for all $u, v \in V$

(v) Associativity: $(u + v) + w = u + (v + w)$ and $\alpha(\beta v) = (\alpha\beta)v$ for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$

(vi) Distributivity: $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)v = \alpha v + \beta v$ for all $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$

We will assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ as these are the most common cases by far, although in general it could be any field.

To justify the notations 0 for the additive identity and $-v$ for the additive inverse of v we must demonstrate their uniqueness.

Proof. If $w, \tilde{w} \in V$ satisfy $v + w = v$ and $v + \tilde{w} = v$ for all $v \in V$, then

$$w = w + \tilde{w} = \tilde{w} + w = \tilde{w}$$

This proves the uniqueness of the additive identity. Similarly, if $w, \tilde{w} \in V$ satisfy $v + w = 0$ and $v + \tilde{w} = 0$, then

$$w = w + 0 = w + v + \tilde{w} = 0 + \tilde{w} = \tilde{w}$$

thus showing the uniqueness of the additive inverse. ☐

We have $0v = 0$ for every $v \in V$ and $\alpha 0 = 0$ for every $\alpha \in \mathbb{F}$.

Proof. If $v \in V$, then

$$v = 1v = (1+0)v = 1v + 0v = v + 0v$$

which implies $0v = 0$. If $\alpha \in \mathbb{F}$, then

$$\alpha v = \alpha(v + 0) = \alpha v + \alpha 0$$

which implies $\alpha 0 = 0$. □

Moreover, $(-1)v = -v$ for every $v \in V$, since

$$v + (-1)v = (1 + (-1))v = 0v = 0$$

and we have shown that additive inverses are unique.

2.1 Subspaces

Vector spaces can contain other vector spaces. If V is a vector space, then $S \subseteq V$ is said to be a **subspace** of V if

(i) $0 \in S$

(ii) S is closed under addition: $u, v \in S$ implies $u + v \in S$

(iii) S is closed under scalar multiplication: $v \in S, \alpha \in \mathbb{F}$ implies $\alpha v \in S$

Note that V is always a subspace of V, as is the trivial vector space which contains only 0.

Proposition 1. *Suppose U and W are subspaces of some vector space. Then $U \cap W$ is a subspace of U and a subspace of W.*

Proof. We only show that $U \cap W$ is a subspace of U; the same result follows for W since $U \cap W = W \cap U$.

(i) Since $0 \in U$ and $0 \in W$ by virtue of their being subspaces, we have $0 \in U \cap W$.

(ii) If $x, y \in U \cap W$, then $x, y \in U$ so $x + y \in U$, and $x, y \in W$ so $x + y \in W$; hence $x + y \in U \cap W$.

(iii) If $v \in U \cap W$ and $\alpha \in \mathbb{F}$, then $v \in U$ so $\alpha v \in U$, and $v \in W$ so $\alpha v \in W$; hence $\alpha v \in U \cap W$.

Thus $U \cap W$ is a subspace of U. □

2.1.1 Sums of subspaces

If U and W are subspaces of V, then their sum is defined as

$$U + W \triangleq \{u + w : u \in U, w \in W\}$$

Proposition 2. *Let U and W be subspaces of V. Then $U + W$ is a subspace of V.*

Proof. (i) Since $0 \in U$ and $0 \in V$, $0 = 0 + 0 \in U + W$.

(ii) If $v_1, v_2 \in U + W$, then there exist $u_1, u_2 \in U$ and $w_1, w_2 \in W$ such that $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$, so by the closure of U and W under addition we have

$$v_1 + v_2 = u_1 + w_1 + u_2 + w_2 = \underbrace{u_1 + u_2}_{\in U} + \underbrace{w_1 + w_2}_{\in W} \in U + W$$

(iii) If $v \in U + W, \alpha \in F$, then there exists $u \in U$ and $w \in W$ such that $v = u + w$, so by the closure of U and W under scalar multiplication we have

$$\alpha v = \alpha(u + w) = \underbrace{\alpha u}_{\in U} + \underbrace{\alpha w}_{\in W} \in U + W$$

Thus $U + W$ is a subspace of V. □

If $U \cap W = \{0\}$, the sum is said to be a **direct sum** and written $U \oplus W$.

Proposition 3. *Suppose U and W are subspaces of some vector space. The following are equivalent:*

(i) The sum $U + W$ is direct, i.e. $U \cap W = \{0\}$.

(ii) The only way to write 0 as a sum $u + w$ where $u \in U, w \in W$ is to take $u = w = 0$.

(iii) Every vector in $U + W$ can be written uniquely as $u + w$ for some $u \in U$ and $w \in W$.

Proof. (i) $\implies$ (ii): Assume (i), and suppose $0 = u + w$ where $u \in U, w \in W$. Then $u = -w$, so $u \in W$ as well, and thus $u \in U \cap W$. By assumption this implies $u = 0$, so $w = 0$ also.

(ii) $\implies$ (iii): Assume (ii), and suppose $v \in U + W$ can be written both as $v = u + w$ and $v = \tilde{u} + \tilde{w}$ where $u, \tilde{u} \in U$ and $w, \tilde{w} \in W$. Then

$$0 = v - v = u + w - (\tilde{u} + \tilde{w}) = \underbrace{u - \tilde{u}}_{\in U} + \underbrace{w - \tilde{w}}_{\in W}$$

so by assumption we must have $u - \tilde{u} = w - \tilde{w} = 0$, i.e. $u = \tilde{u}$ and $w = \tilde{w}$.

(iii) $\implies$ (i): Assume (iii), and suppose $v \in U \cap W$, so that $v \in U$ and $v \in W$, nothing that $-v \in W$ also. Now $0 = v + (-v)$ and $0 = 0 + 0$ both write 0 in the form $u + w$ where $u \in U, w \in W$, so by the assumed uniqueness of this decomposition we conclude that $v = 0$. Thus $U \cap W \subseteq \{0\}$. It is also clear that $\{0\} \subseteq U \cap W$ since U and W are both subspaces, so $U \cap W = \{0\}$. □

2.2 Span

A **linear combination** of vectors $v_1, \dots, v_n \in V$ is an expression of the form

$$\alpha_1 v_1 + \cdots + \alpha_n v_n$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{F}$. The **span** of a set of vectors is the set of all possible linear combinations of these:

$$\operatorname{span}\{v_1, \dots, v_n\} \triangleq \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \alpha_1, \dots, \alpha_n \in \mathbb{F}\}$$

We also define the special case $\operatorname{span}\{\} = \{0\}$. Spans of sets of vectors form subspaces.

Proposition 4. *If $v_1, \dots, v_n \in V$, then $\operatorname{span}\{v_1, \dots, v_n\}$ is a subspace of V.*

Proof. Let $S = \text{span}\{v_1, \dots, v_n\}$. If $n = 0$, we have $S = \text{span}\{\} = \{0\}$, which is a subspace of V, so assume hereafter that $n > 0$.

(i) Since $\{v_1, \dots, v_n\}$ is non-empty (as $n \geq 1$), we have $0 = 0v_1 + \dots + 0v_n \in S$.

(ii) If $x, y \in S$, we can write $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ and $y = \beta_1 v_1 + \dots + \beta_n v_n$ for some $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{F}$. Then

$$x + y = \alpha_1 v_1 + \dots + \alpha_n v_n + \beta_1 v_1 + \dots + \beta_n v_n = (\alpha_1 + \beta_1)v_1 + \dots + (\alpha_n + \beta_n)v_n \in S$$

(iii) If $x \in S, \beta \in \mathbb{F}$, we can write $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{F}$. Then

$$\beta x = \beta(\alpha_1 v_1 + \dots + \alpha_n v_n) = (\beta\alpha_1)v_1 + \dots + (\beta\alpha_n)v_n \in S$$

Thus S is a subspace of V. □

Adding a vector which lies in the span of a set of vectors to that set does not expand the span in any way.

Proposition 5. *If $w \in \text{span}\{v_1, \dots, v_n\}$, then $\text{span}\{v_1, \dots, v_n, w\} = \text{span}\{v_1, \dots, v_n\}$.*

Proof. Suppose $w \in \text{span}\{v_1, \dots, v_n\}$. It is clear that $\text{span}\{v_1, \dots, v_n\} \subseteq \text{span}\{v_1, \dots, v_n, w\}$: if $u \in \text{span}\{v_1, \dots, v_n\}$ then it can be written $u = \alpha_1 v_1 + \dots + \alpha_n v_n$, but then $u = \alpha_1 v_1 + \dots + \alpha_n v_n + 0w$, so $u \in \text{span}\{v_1, \dots, v_n, w\}$.

To prove the other direction, write $w = \beta_1 v_1 + \dots + \beta_n v_n$. Then for any $u \in \text{span}\{v_1, \dots, v_m, w\}$, there exist $\alpha_1, \dots, \alpha_n, \alpha_{n+1}$ such that $u = \alpha_1 v_1 + \dots + \alpha_n v_n + \alpha_{n+1} w$, and

$$\begin{aligned} u &= \alpha_1 v_1 + \dots + \alpha_n v_n + \alpha_{n+1}(\beta_1 v_1 + \dots + \beta_n v_n) \\ &= (\alpha_1 + \alpha_{n+1}\beta_1)v_1 + \dots + (\alpha_n + \alpha_{n+1}\beta_n)v_n \end{aligned}$$

so $u \in \text{span}\{v_1, \dots, v_m\}$. Hence $\text{span}\{v_1, \dots, v_n, w\} \subseteq \text{span}\{v_1, \dots, v_n\}$ and the claim follows. □

2.3 Linear independence

A set of vectors $v_1, \dots, v_n \in V$ is said to be **linearly independent** if

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \quad \text{implies} \quad \alpha_1 = \dots = \alpha_n = 0.$$

Otherwise, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, not all zero, such that $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$, and $v_1, \dots, v_n$ are said to be **linearly dependent**. Note that in the case of linear dependence, (at least) one vector can be written as a linear combination of the other vectors in the set: if $\alpha_j \neq 0$, then

$$v_j = -\frac{1}{\alpha_j} \sum_{k \neq j}^{n} \alpha_k v_k$$

This implies that $v_j \in \text{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$.

2.4 Basis

If a set of vectors is linearly independent and its span is the whole of V, those vectors are said to be a **basis** for V. One of the most important properties of bases is that they provide unique representations for every vector in the space they span.

Proposition 6. *A set of vectors* $v_1, \dots, v_n \in V$ *is a basis for* V *if and only if every vector in* V *can be written uniquely as a linear combination of* $v_1, \dots, v_n$.

Proof. Suppose $v_1, \dots, v_n$ is a basis for V. Then these span V, so if $w \in V$ then there exist $\alpha_1, \dots, \alpha_n$ such that $w = \alpha_1 v_1 + \dots + \alpha_n v_n$. Furthermore, if $w = \tilde{\alpha}_1 v_1 + \dots + \tilde{\alpha}_n v_n$ also, then

$$\begin{aligned} 0 &= w - w \\ &= \alpha_1 v_1 + \dots + \alpha_n v_n - (\tilde{\alpha}_1 v_1 + \dots + \tilde{\alpha}_n v_n) \\ &= (\alpha_1 - \tilde{\alpha}_1) v_n + \dots + (\alpha_n - \tilde{\alpha}_n) v_n \end{aligned}$$

Since $v_1, \dots, v_n$ are linearly independent, this implies $\alpha_j - \tilde{\alpha}_j = 0$, i.e. $\alpha_j = \tilde{\alpha}_j$, for all $j = 1, \dots, n$.

Conversely, suppose that every vector in V can be written uniquely as a linear combination of $v_1, \dots, v_n$. The existence part of this assumption clearly implies that $v_1, \dots, v_n$ span V. To show that $v_1, \dots, v_n$ are linearly independent, suppose

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0$$

and observe that since

$$0 v_1 + \dots + 0 v_n = 0$$

the uniqueness of the representation of 0 as a linear combination of $v_1, \dots, v_n$ implies that $\alpha_1 = \dots = \alpha_n = 0$. □

If a vector space is spanned by a finite number of vectors, it is said to be **finite-dimensional**. Otherwise it is **infinite-dimensional**. The number of vectors in a basis for a finite-dimensional vector space V is called the **dimension** of V and denoted $\dim V$.

Proposition 7. *If* $v_1, \dots, v_n \in V$ *are not all zero, then there exists a subset of* $\{v_1, \dots, v_n\}$ *which is linearly independent and whose span equals* $\operatorname{span}\{v_1, \dots, v_n\}$.

Proof. If $v_1, \dots, v_n$ are linearly independent, then we are done. Otherwise, pick v_j which is in $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$. Since $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_n\}$, we can drop v_j from this list without changing the span. Repeat this process until the resulting set is linearly independent. □

As a corollary, we can establish an important fact: every finite-dimensional vector space has a basis.

Proof. Let V be a finite-dimensional vector space. By definition, this means that there exist $v_1, \dots, v_n$ such that $V = \operatorname{span}\{v_1, \dots, v_n\}$. By the previous result, there exists a linearly independent subset of $\{v_1, \dots, v_n\}$ whose span is V; this subset is a basis for V. □

Proposition 8. *A linearly independent set of vectors in* V *can be extended to a basis for* V.

Proof. Suppose $v_1, \dots, v_n \in V$ are linearly independent. Let $w_1, \dots, w_m$ be a basis for V. Then $v_1, \dots, v_n, w_1, \dots, w_m$ spans V, so there exists a subset of this set which is linearly independent and also spans V. Moreover, by inspecting the process by which this linearly independent subset is produced, we see that none of $v_1, \dots, v_n$ need be dropped because they are linearly independent. Hence the resulting linearly independent set, which is basis for V, can be chosen to contain $v_1, \dots, v_n$. □

The dimensions of sums of subspaces obey a friendly relationship:

Proposition 9. *If U and W are subspaces of some vector space, then*

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

Proof. Let $v_1, \dots, v_k$ be a basis for $U \cap W$, and extend this to bases $v_1, \dots, v_k, u_1, \dots, u_m$ for U and $v_1, \dots, v_k, w_1, \dots, w_n$ for W. We claim that $v_1, \dots, v_k, u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for $U+W$.

If $v \in U+W$, then $v = u + w$ for some $u \in U, w \in W$. Then since $u = \alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha_{k+1} u_1 \cdots + \alpha_{k+m} u_m$ for some $\alpha_1, \dots, \alpha_{k+m} \in \mathbb{F}$ and $w = \beta_1 v_1 + \cdots + \beta_k v_k + \beta_{k+1} w_1 \cdots + \beta_{k+n} w_n$ for some $\beta_1, \dots, \beta_{k+n} \in \mathbb{F}$, we have

$$\begin{aligned} v &= u + w \\ &= \alpha_1 v_1 + \cdots + \alpha_k v_k + \alpha_{k+1} u_1 \cdots + \alpha_{k+m} u_m + \beta_1 v_1 + \cdots + \beta_k v_k + \beta_{k+1} w_1 \cdots + \beta_{k+n} w_n \\ &= (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_k + \beta_k) v_k + \alpha_{k+1} u_1 \cdots + \alpha_{k+m} u_m + \beta_{k+1} w_1 \cdots + \beta_{k+n} w_n \end{aligned}$$

so $v \in \operatorname{span}\{v_1, \dots, v_k, u_1, \dots, u_m, w_1, \dots, w_n\}$. Moreover, the uniqueness of the expansion of v in this set follows from the uniqueness of the expansions of u and w.

Thus $v_1, \dots, v_k, u_1, \dots, u_m, w_1, \dots, w_n$ is a basis for $U+W$ as claimed, so

$$\dim U + \dim W - \dim(U \cap W) = (k+m) + (k+n) - k = k + m + n = \dim(U+W)$$

as we set out to show. □

It follows immediately that

$$\dim(U \oplus W) = \dim U + \dim W$$

since $\dim(U \cap W) = \dim\{0\} = 0$ if the sum is direct.

3 Linear maps

A **linear map** is a function $T : V \to W$, where V and W are vector spaces over $\mathbb{F}$, that satisfies

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

(ii) $T(\alpha v) = \alpha T(v)$ for all $v \in V, \alpha \in \mathbb{F}$

The standard notational convention for linear maps (which we will follow hereafter) is to drop unnecessary parentheses, writing Tv rather than $T(v)$ if there is no risk of ambiguity, and denote composition of linear maps by ST rather than the usual $S \circ T$.

The identity map on V, which sends each $v \in V$ to v, is denoted I_V, or just I if the vector space V is clear from context. Note that all linear maps (not just the identity) send zero to zero.

Proof. For any $v \in V$ we have

$$Tv = T(v + 0_V) = Tv + T0_V$$

so by subtracting Tv from both sides we obtain $T0_V = 0_W$. □

One useful fact regarding linear maps (both conceptually and for proofs) is that they are uniquely determined by their action on a basis.

Proposition 10. *Let $v_1, \dots, v_n$ be a basis for a vector space V, and $w_1, \dots, w_n$ be arbitrary vectors in a vector space W. Then there exists a unique linear map $T : V \to W$ such that*

$$Tv_j = w_j$$

for all $j = 1, \dots, n$.

Proof. Define

$$\begin{aligned} T : V &\to W \\ \alpha_1 v_1 + \cdots + \alpha_n v_n &\mapsto \alpha_1 w_1 + \cdots + \alpha_n w_n \end{aligned}$$

This function is well-defined because every vector in V can be expressed uniquely as a linear combination of $v_1, \dots, v_n$. Then for any $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $y = \beta_1 v_1 + \cdots + \beta_n v_n$ in V,

$$\begin{aligned} T(x+y) &= T(\alpha_1 v_1 + \cdots + \alpha_n v_n + \beta_1 v_1 + \cdots + \beta_n v_n) \\ &= T((\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n + \beta_n)v_n) \\ &= (\alpha_1 + \beta_1)w_1 + \cdots + (\alpha_n + \beta_n)w_n \\ &= \alpha_1 w_1 + \cdots + \alpha_n w_n + \beta_1 w_1 + \cdots + \beta_n w_n \\ &= Tx + Ty \end{aligned}$$

and if $\gamma \in \mathbb{F}$ then

$$\begin{aligned} T(\gamma x) &= T(\gamma(\alpha_1 v_1 + \cdots + \alpha_n v_n)) \\ &= T((\gamma\alpha_1)v_1 + \cdots + (\gamma\alpha_n)v_n) \\ &= (\gamma\alpha_1)w_1 + \cdots + (\gamma\alpha_n)w_n \\ &= \gamma(\alpha_1 w_1 + \cdots + \alpha_n w_n) \\ &= \gamma Tx \end{aligned}$$

so T is a linear map. Now observe that, by construction,

$$\begin{aligned} Tv_j &= T(0v_1 + \cdots + 0v_{j-1} + 1v_j + 0v_{j+1} + \cdots + 0v_n) \\ &= 0w_1 + \cdots + 0w_{j-1} + 1w_j + 0w_{j+1} + \cdots + 0w_n \\ &= w_j \end{aligned}$$

as desired. Towards uniqueness, suppose $\tilde{T}$ is a linear map which satisfies $\tilde{T}v_j = w_j$ for $j = 1, \dots, n$. Any $v \in V$ can be written uniquely as $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$, and

$$\tilde{T}(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 \tilde{T} v_1 + \cdots + \alpha_n \tilde{T} v_n = \alpha_1 w_1 + \cdots + \alpha_n w_n$$

by the linearity of $\tilde{T}$. Hence $\tilde{T} = T$. □

3.1 Isomorphisms

Observe that the definition of a linear map is suited to reflect the structure of vector spaces, since it preserves vector spaces' two main operations, addition and scalar multiplication. In algebraic terms, a linear map is said to be a **homomorphism** of vector spaces. An invertible homomorphism where the inverse is also a homomorphism is called an **isomorphism**. If there exists an isomorphism from V to W, then V and W are said to be **isomorphic**, and we write $V \cong W$. Isomorphic vector spaces are essentially "the same" in terms of their algebraic structure. It is an interesting fact that finite-dimensional vector spaces of the same dimension over the same field are always isomorphic.

Proof. Let V and W be finite-dimensional vector spaces over $\mathbb{F}$, with $n \triangleq \dim V = \dim W$. We will show that $V \cong W$ by exhibiting an isomorphism from V to W. To this end, let $v_1, \dots, v_n$ and $w_1, \dots, w_n$ be bases for V and W, respectively. Then there exist (unique) linear maps $T : V \to W$ and $S : W \to V$ satisfying $Tv_j = w_j$ and $Sw_j = v_j$ for all $j = 1, \dots, n$. Now clearly $STv_j = Sw_j = v_j$ for each j, so $ST = I_V$. Similarly, $TSw_j = Tv_j = w_j$ for each j, so $TS = I_W$. Hence $S = T^{-1}$, so T is an isomorphism, and hence $V \cong W$. □

3.2 Algebra of linear maps

Linear maps can be added and scaled to produce new linear maps. That is, if S and T are linear maps from V into W, and $\alpha \in \mathbb{F}$, it is straightforward to verify that $S + T$ and αT are linear maps from V into W. Since addition and scaling of functions satisfy the usual commutativity/associativity/distributivity rules, the set of linear maps from V into W is also a vector space over $\mathbb{F}$; we denote this space by $L(V, W)$. The additive identity here is the zero map which sends every $v \in V$ to 0_W.

The composition of linear maps is also a linear map.

Proposition 11. *Let U, V, and W be vector spaces over a common field $\mathbb{F}$, and suppose $S : V \to W$ and $T : U \to V$ are linear maps. Then the composition $ST : U \to W$ is also a linear map.*

Proof. For any $u, v \in U$,

$$(ST)(u + v) = S(T(u + v)) = S(Tu + Tv) = STu + STv$$

and furthermore if $\alpha \in \mathbb{F}$ then

$$(ST)(\alpha v) = S(T(\alpha v)) = S(\alpha Tv) = \alpha STv$$

so ST is linear. □

The identity map is a "multiplicative" identity here, as $TI_V = I_W T = T$ for any $T : V \to W$.

3.3 Nullspace

If $T : V \to W$ is a linear map, the **nullspace**[1] of T is the set of all vectors in V that get mapped to zero:

$$\text{null}(T) \triangleq \{v \in V : Tv = 0_W\}$$

Proposition 12. *If $T : V \to W$ is a linear map, then* null(T) *is a subspace of V.*

Proof. (i) We have already seen that $T(0_V) = 0_W$, so $0_V \in \text{null}(T)$.

(ii) If $u, v \in \text{null}(T)$, then

$$T(u + v) = Tu + Tv = 0_W + 0_W = 0_W$$

so $u + v \in \text{null}(T)$.

[1] It is sometimes called the **kernel** by algebraists, but we eschew this terminology because the word "kernel" has another meaning in machine learning.

(iii) If $v \in \text{null}(T), \alpha \in \mathbb{F}$, then
$$T(\alpha v) = \alpha T v = \alpha 0_W = 0_W$$
so $\alpha v \in \text{null}(T)$.

Thus null(T) is a subspace of V. □

Proposition 13. *A linear map $T : V \to W$ is injective if and only if* $\text{null}(T) = \{0_V\}$.

Proof. Assume T is injective. We have seen that $T0_V = 0_W$, so $0_V \in \text{null}(T)$, and moreover if $Tv = 0_W$ the injectivity of T implies $v = 0_V$, so $\text{null}(T) = \{0_V\}$.

Conversely, assume $\text{null}(T) = \{0_V\}$, and suppose $Tu = Tv$ for some $u, v \in V$. Then
$$0 = Tu - Tv = T(u - v)$$
so $u - v \in \text{null}(T)$, which by assumption implies $u - v = 0$, i.e. $u = v$. Hence T is injective. □

3.4 Range

The **range** of T is (as for any function) the set of all possible outputs of T:
$$\text{range}(T) \triangleq \{Tv : v \in V\}$$

Proposition 14. *If $T : V \to W$ is a linear map, then* range(T) *is a subspace of W.*

Proof. (i) We have already seen that $T(0_V) = 0_W$, so $0_W \in \text{range}(T)$.

(ii) If $w, z \in \text{range}(T)$, there exist $u, v \in V$ such that $w = Tu$ and $z = Tv$. Then
$$T(u + v) = Tu + Tv = w + z$$
so $w + z \in \text{range}(T)$.

(iii) If $w \in \text{range}(T), \alpha \in \mathbb{F}$, there exists $v \in V$ such that $w = Tv$. Then
$$T(\alpha v) = \alpha T v = \alpha w$$
so $\alpha w \in \text{range}(T)$.

Thus range(T) is a subspace of W. □

4 Eigenvalues and eigenvectors

In the special case where the domain and codomain of a linear map are the same, certain vectors may have the special property that the map simply scales them. If $T : V \to V$ is a linear map and $v \in V$ is nonzero, then v is said to be an **eigenvector** of T with corresponding **eigenvalue** $\lambda \in \mathbb{F}$ if
$$Tv = \lambda v$$
or equivalently, if
$$v \in \text{null}(T - \lambda I)$$
This second definition makes it clear that the set of eigenvectors corresponding to a given eigenvalue (along with 0) is a vector space. This vector space is called the **eigenspace** of T associated with λ.