Online Appendix to "Motivational Ratings"

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This appendix contains supplementary material and proofs omitted from the main text. References to this Online Appendix are preceded by the letters "OA." Other references are to the main text.

Contents

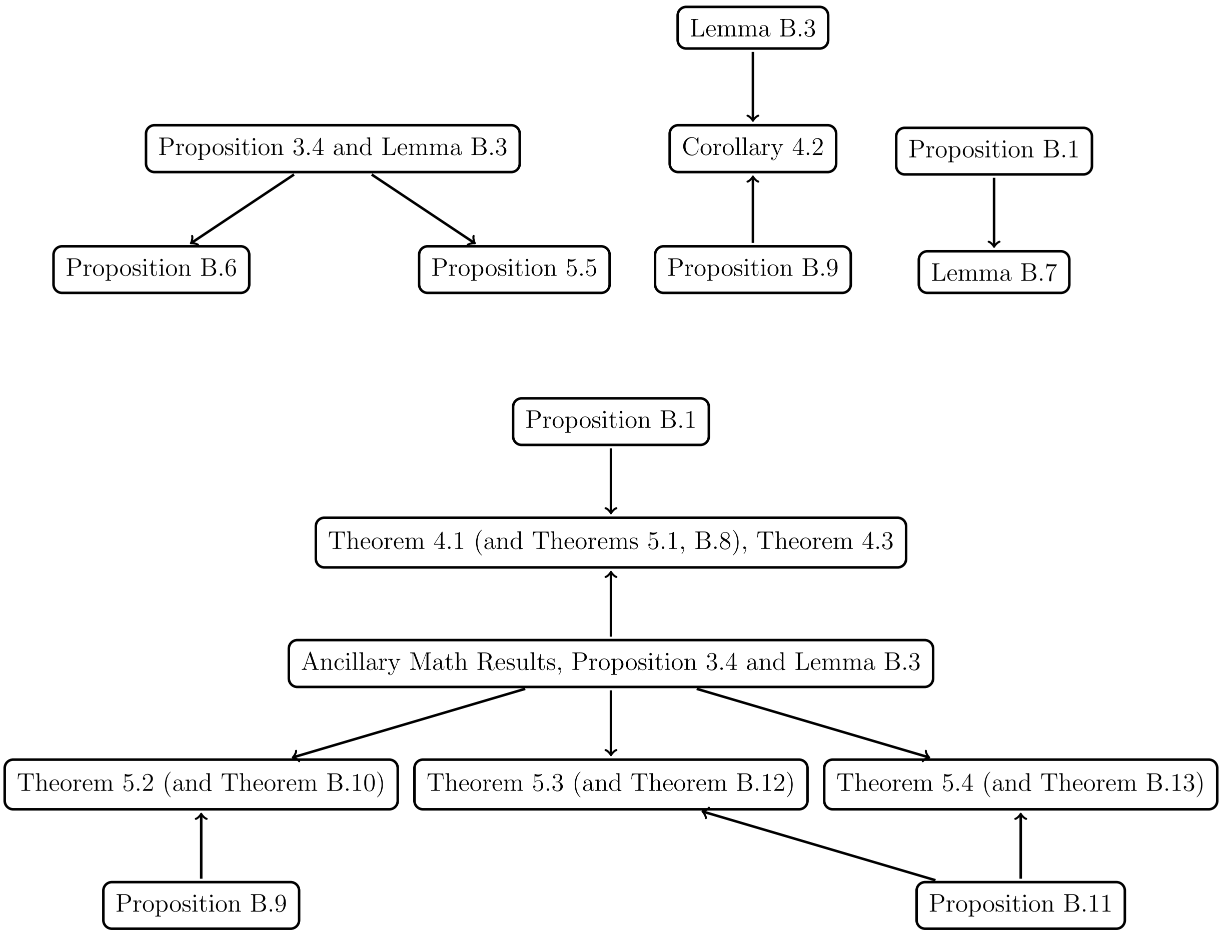

Figure 1: Dependencies between the different results.

Part I

Complements for Section 2

OA.1 Additional Details on the Computations

In this section we provide the details on the computations of Section 2 in the main text.

First, we derive the equilibrium effort for the standard model, the first example of Section 2. Equation (3) (in the main text) implies that, if the worker exerts effort A_t at round t, then the expected market belief, from the viewpoint of the worker, is

$$\mathbf{E}[\mu_t] = \frac{\Sigma}{1+(t-1)\Sigma}\sum_{s=1}^{t-1}(A_s - A_s^*). \tag{OA.1}$$

The worker chooses each effort level A_t, $t = 1, \ldots, T$ so as to maximize Equation (2), which after plugging in (OA.1) is equal to

$$\begin{aligned}
\sum_{t=1}^{T}\delta^{t-1}\mathbf{E}[\mu_t - c(A_t)] &= -\sum_{t=1}^{T}\delta^{t-1}\frac{A_t^2}{2} + \sum_{t=1}^{T}\sum_{s=1}^{t-1}\delta^{t-1}\frac{\Sigma}{1+(t-1)\Sigma}(A_s - A_s^*)\\
&= -\sum_{t=1}^{T}\delta^{t-1}\frac{A_t^2}{2} + \sum_{t=1}^{T}\sum_{s=t+1}^{T}\delta^{s-1}\frac{\Sigma}{1+(s-1)\Sigma}(A_t - A_t^*).
\end{aligned}$$

Hence, maximizing (3) yields a unique optimal effort level A_t at each round t, which is the maximizer of

$$-\delta^{t-1}\frac{A_t^2}{2} + \sum_{s=t+1}^{T}\delta^{s-1}\frac{\Sigma}{1+(s-1)\Sigma}(A_t - A_t^*).$$

Hence

$$A_t = \sum_{s=t+1}^{T}\delta^{s-t}\frac{\Sigma}{1+(s-1)\Sigma},$$

which corresponds to Equation (4).

Next, we derive the equilibrium effort for the two-signal two-period cases for the second and third examples of Section 2. Recall that in this variant of the standard model,

$$\begin{aligned}
S_1^{\mathrm{I}} &= \alpha^{\mathrm{I}}A_1 + \beta^{\mathrm{I}}\theta + \varepsilon_1^{\mathrm{I}},\\
S_1^{\mathrm{II}} &= \alpha^{\mathrm{II}}A_1 + \beta^{\mathrm{II}}\theta + \varepsilon_1^{\mathrm{II}},
\end{aligned}$$

and

$$Y_2 = \lambda S_1^{\mathrm{I}} + (1-\lambda)S_1^{\mathrm{II}}.$$

As in the standard model the worker exerts no effort in the second round ($A_2 = 0$). The equilibrium effort in the first round is determined by the value of the rating at the start of round 2: the worker chooses effort A_1 to maximize

$$\sum_{t=1}^{2} \delta^{t-1} \mathbf{E}[\mu_t - c(A_t)] = \delta \mathbf{E}[\mu_2] - c(A_1), \tag{OA.2}$$

where the expectation reflects the worker's viewpoint, *i.e.*, it assumes the distribution over signals obtained under effort A_t. Using the standard projection formulas for the multivariate normal distribution, we get

$$\begin{aligned} \mu_2 = \mathbf{E}[\theta \mid Y_2] &= \mathbf{E}[\theta] + \frac{\mathbf{Cov}[\theta, Y_2]}{\mathbf{Var}[Y_2]} (Y_2 - \mathbf{E}[Y_2]), \\ &= \frac{\mathbf{Cov}[\theta, Y_2]}{\mathbf{Var}[Y_2]} \left(Y_2 - (\lambda\alpha^{\mathrm{I}} + (1-\lambda)\alpha^{\mathrm{II}})A_1^*\right), \end{aligned}$$

where this time the expectation reflects the market's viewpoint, *i.e.*, it assumes that the worker exerts effort A_1^* during the first round, as opposed to A_1. The market's belief may thus differ from the worker's belief off equilibrium when $A_1^* \neq A_1$. We also have

$$\begin{aligned} \mathbf{Cov}[\theta, Y_2] &= \lambda\beta^{\mathrm{I}}\Sigma + (1-\lambda)\beta^{\mathrm{II}}\Sigma, \\ \mathbf{Var}[Y_2] &= (\lambda\beta^{\mathrm{I}} + (1-\lambda)\beta^{\mathrm{II}})^2\Sigma + \lambda^2 + (1-\lambda)^2. \end{aligned}$$

Covariance and variance are also taken with respect to the market's viewpoint, but their expression remains the same when taken with respect to the worker's viewpoint, so it is irrelevant. Overall, we can rewrite Equation (OA.2) as

$$\delta \frac{\lambda\beta^{\mathrm{I}}\Sigma + (1-\lambda)\beta^{\mathrm{II}}\Sigma}{(\lambda\beta^{\mathrm{I}} + (1-\lambda)\beta^{\mathrm{II}})^2\Sigma + \lambda^2 + (1-\lambda)^2} (\lambda\alpha^{\mathrm{I}} + (1-\lambda)\alpha^{\mathrm{II}}) (A_1 - A_1^*) - \frac{A_1^2}{2},$$

and so the effort level A_1 that maximizes (OA.2) is

$$A_1 = \delta \frac{\lambda\beta^{\mathrm{I}}\Sigma + (1-\lambda)\beta^{\mathrm{II}}\Sigma}{(\lambda\beta^{\mathrm{I}} + (1-\lambda)\beta^{\mathrm{II}})^2\Sigma + \lambda^2 + (1-\lambda)^2} (\lambda\alpha^{\mathrm{I}} + (1-\lambda)\alpha^{\mathrm{II}}), \tag{OA.3}$$

which corresponds to Equation (5) when $(\alpha^{\mathrm{I}}, \beta^{\mathrm{I}}) = (1, 0)$ and $(\alpha^{\mathrm{II}}, \beta^{\mathrm{II}}) = (0, 1)$. The optimal weights of Equations (6) and (7) are obtained by maximizing the equilibrium effort given by (OA.3) for the two cases $(\alpha^{\mathrm{I}}, \beta^{\mathrm{I}}) = (1, 0)$, $(\alpha^{\mathrm{II}}, \beta^{\mathrm{II}}) = (0, 1)$ and $(\alpha^{\mathrm{I}}, \beta^{\mathrm{I}}) = (\alpha^{\mathrm{I}}, 1)$, $(\alpha^{\mathrm{I}}, \beta^{\mathrm{II}}) = (1, 1)$.

We now derive the equilibrium effort for the one-signal three-period case, which is the fourth example of Section 2. At the first round, the rating is irrelevant and $\mu_1 = 0$. At the second round, the rating chosen is $Y_2 = X_1$ and the market belief is

$$\begin{aligned}\mu_2 = \mathbf{E}[\theta \mid X_1] = \mathbf{E}[\theta] + \frac{\mathbf{Cov}[\theta, X_1]}{\mathbf{Var}[X_1]}(X_1 - \mathbf{E}[X_1]),\\ = \frac{\mathbf{Cov}[\theta, X_1]}{\mathbf{Var}[X_1]}(X_1 - A_1^*),\\ = \frac{\Sigma}{\Sigma+1}(X_1 - A_1^*),\end{aligned}$$

by another application of the projection formulas for the multivariate normal distribution, where the expectations are taken from the market's perspective and where we note that

$$\mathbf{Cov}[\theta, X_1] = \Sigma, \quad \text{and} \quad \mathbf{Var}[X_1] = \Sigma + 1.$$

At the third round, the rating is $Y_3 = \lambda X_1 + (1-\lambda)X_2$ and the market belief is

$$\begin{aligned}\mu_3 &= \frac{\mathbf{Cov}[\theta, Y_3]}{\mathbf{Var}[Y_3]}(Y_3 - \lambda A_1^* - (1-\lambda)A_2^*),\\ &= \frac{\Sigma}{\lambda^2 + (1-\lambda)^2 + \Sigma}(Y_3 - \lambda A_1^* - (1-\lambda)A_2^*),\end{aligned}$$

where we note that

$$\mathbf{Cov}[\theta, Y_3] = \Sigma, \quad \text{and} \quad \mathbf{Var}[Y_3] = \lambda^2 + (1-\lambda)^2 + \Sigma.$$

Therefore, at round 2 and 3, the expected market beliefs from the worker's viewpoint are respectively

$$\mathbf{E}[\mu_2] = \frac{\Sigma}{\Sigma+1}(A_1 - A_1^*), \tag{OA.4}$$

and

$$\mathbf{E}[\mu_3] = \frac{\Sigma}{\lambda^2 + (1-\lambda)^2 + \Sigma}(\lambda A_1 + (1-\lambda)A_2 - \lambda A_1^* - (1-\lambda)A_2^*). \tag{OA.5}$$

The worker chooses effort $A_3 = 0$ at round 3, and chooses efforts A_1 and A_2, at round 1 and 2 respectively, to maximize

$$\sum_{t=1}^{3} \delta^{t-1}\mathbf{E}[\mu_t - c(A_t)] = \delta\mathbf{E}[\mu_2] + \delta^2\mathbf{E}[\mu_3] - c(A_1) - \delta c(A_2). \tag{OA.6}$$

After plugging Equations (OA.4) and (OA.5) into Equation (OA.6), we get the optimal effort levels

$$A_1 = \delta\Sigma\left(\frac{\delta\lambda}{2(\lambda-1)\lambda + \Sigma + 1} + \frac{1}{\Sigma+1}\right),$$

and

$$A_2 = -\frac{\delta(\lambda-1)\Sigma}{2(\lambda-1)\lambda+\Sigma+1}.$$

Choosing λ that maximizes $A_1 + A_2$ yields

$$\lambda = \frac{\sqrt{2}\sqrt{\delta^2+(\delta-1)^2\Sigma+1}-2}{2(\delta-1)},$$

which after simplification is equal to $\Lambda(\delta)$ defined in Equation (7).

Finally, we consider the last example of Section 2, with T rounds, and a rating linear in output: at the start of round t, the market observes

$$Y_t = \sum_{s=1}^{t-1} u_{s,t} X_s.$$

From the now usual projection formulas for multivariate normal distributions, the market belief about the worker's ability is then

$$\mu_t = \mathbf{E}[\theta \mid Y_t] = \frac{\mathbf{Cov}[\theta, Y_t]}{\mathbf{Var}[Y_t]}\left(Y_t - \mathbf{E}[Y_t]\right),$$

where the expectation is taken according to the market's conjecture of the worker's effort A^*, and with

$$\mathbf{Cov}[\theta, Y_t] = \Sigma \sum_{k=1}^{t-1} u_{k,t},$$

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{t-1} u_{k,t}^2 + \Sigma \sum_{i,j=1}^{t-1} u_{i,t}u_{j,t}.$$

Hence,

$$\mu_t = \frac{\sum_{k=1}^{t-1} u_{k,t}}{\Sigma^{-1}\sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t}u_{j,t}}\left(Y_t - \sum_{k=1}^{t-1} u_{k,t}A_k^*\right).$$

The worker chooses his efforts A_t, $t = 1, \ldots, T$ to maximize

$$\mathbf{E}\left[\sum_{t=1}^{T} \delta^{t-1}\left(\mu_t - c(A_t)\right)\right]$$
$$= \sum_{t=1}^{T} \delta^{t-1}\left(\frac{\sum_{k=1}^{t-1} u_{k,t}}{\Sigma^{-1}\sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t}u_{j,t}}\left(\sum_{s=1}^{t-1} u_{k,t}(A_k - A_k^*)\right) - \frac{A_t^2}{2}\right).$$

The first-order condition yields

$$A_s = \sum_{t=s+1}^{T} \frac{\delta^{t-s} u_{s,t} \sum_{k=1}^{t-1} u_{kt}}{\Sigma^{-1} \sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t} u_{j,t}},$$

and the second-order condition is always satisfied. Finding the parameters $u_{s,t}$, $s < t$ so as to maximize aggregate equilibrium effort

$$\sum_{s=1}^{T} A_s,$$

and so maximizing

$$\sum_{t=1}^{T} \sum_{s=1}^{t-1} \frac{\delta^{t-s} u_{s,t} \sum_{k=1}^{t-1} u_{k,t}}{\Sigma^{-1} \sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t} u_{j,t}} = \sum_{t=1}^{T} \frac{\left(\sum_{k=1}^{t-1} \delta^{t-s} u_{k,t}\right) \cdot \left(\sum_{k=1}^{t-1} u_{k,t}\right)}{\Sigma^{-1} \sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t} u_{j,t}},$$

is equivalent to finding, for every t, the parameters $u_{s,t}$, $s < t$ that maximize

$$\frac{\left(\sum_{k=1}^{t-1} \delta^{t-k} u_{k,t}\right) \cdot \left(\sum_{k=1}^{t-1} u_{k,t}\right)}{\Sigma^{-1} \sum_{k=1}^{t-1} u_{k,t}^2 + \sum_{i,j=1}^{t-1} u_{i,t} u_{j,t}}.$$

Observe that this last expression is more concisely expressed as

$$\sqrt{\frac{\Sigma}{\mathbf{Var}[Y_t]}}\, \mathbf{Corr}[\theta, Y_t] \sum_{s=1}^{t-1} \delta^{t-s} u_{s,t},$$

which corresponds to Equation (9) of the main text.

Part II
Complements for Section 3

OA.2 Proof of Lemma 3.2

1. If the wage satisfies the zero-profit condition, then the worker who chooses effort strategy A makes (*ex ante*) payoff

$$\mathbf{E}\left[\int_0^\infty (A_t^* + \mu_t - c(A_t))\, e^{-rt}\,\mathrm{d}t\right],$$

where A^* denotes the market conjectured effort level. The worker has no impact on A^*, which may only depend on time. Thus, the worker's strategy is optimal if, and only if, it maximizes

$$\mathbf{E}\left[\int_0^\infty (\mu_t - c(A_t))\, e^{-rt}\,\mathrm{d}t\right].$$

2. Letting $\mathcal{M}' = \{\mathcal{M}'_t\}_{t\geq 0}$ and $\mathcal{M}'_t = \sigma(\mu_t)$, we have $\mathbf{E}^*[\theta_t|\mathcal{M}'_t] = \mathbf{E}^*[\theta_t|\mu_t] = \mu_t$, hence for a given conjectured effort level A^*, the market's transfers and the worker's best response are the same under both information structures $\mathcal{M}$ and $\mathcal{M}'$.

OA.3 Proofs of Proposition 3.4 and Lemma B.3

We prove the existence and uniqueness of the equilibrium, and give the closed-form expression of the equilibrium action in the stationary case. We work directly under the multisignal framework of Section 5, which encompasses the baseline model of Section 3.

Let Y by a linear rating, not necessarily stationary. As Y is linear, then the market belief μ is also a linear rating, by the projection formula for jointly Gaussian variables, and we can write

$$\mu_t = \sum_k \int_{s\leq t} w_{k,s,t}\,(\mathrm{d}S_{k,s} - \alpha_k A_s^*\,\mathrm{d}s)\,,$$

where A^* is the effort level conjectured by the market.

We prove that, given the (unique) wage that satisfies the zero-profit condition, there exists a (up to measure zero sets) unique optimal effort strategy for the worker, pinned down by a first-order condition. This, in turn, yields existence of a unique equilibrium.

Let us fix the wage process π that satisfies the zero-profit condition, and suppose that the worker follows effort strategy A. The worker's time-0 (*ex post*) payoff is then

$$\int_0^\infty (A_t^* + \mu_t - c(A_t))\, e^{-rt}\,\mathrm{d}t. \tag{OA.7}$$

Maximizing the worker's *ex ante* payoff is equivalent to maximizing the *ex post* payoff (up to null events). Hence, we seek conditions on A that characterize when it is a maximizer of (OA.7). Thus, as

$$\mathrm{d}S_{k,s} = (\alpha_k A_s + \beta_k \theta_s)\,\mathrm{d}s + \sigma_k\,\mathrm{d}Z_{k,s},$$

maximizing (OA.7) is equivalent to maximizing

$$\int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k w_{k,s,t} A_s e^{-rt}\,\mathrm{d}s\,\mathrm{d}t - \int_0^\infty c(A_t)e^{-rt}\,\mathrm{d}t. \tag{OA.8}$$

Let us rewrite

$$\int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k w_{k,s,t} A_s e^{-rt}\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty \int_s^\infty \sum_{k=1}^K \alpha_k w_{k,s,t} A_s e^{-rt}\,\mathrm{d}t\,\mathrm{d}s.$$

Maximizing (OA.8) is then the same as maximizing

$$\int_0^\infty \int_s^\infty \sum_{k=1}^K \alpha_k w_{k,s,t} A_s e^{-rt}\,\mathrm{d}t\,\mathrm{d}s - \int_0^\infty c(A_t)e^{-rt}\,\mathrm{d}t,$$

which is the same as maximizing

$$\left(\int_s^\infty \sum_{k=1}^K \alpha_k w_{k,s,t} e^{-rt}\,\mathrm{d}t\right) A_s - c(A_s)e^{-rs},$$

for (almost) every s. By strict convexity of the worker's cost, (OA.8), and thus (OA.7), is maximized if, and only if, for (almost) every t,

$$c'(A_t) = \int_s^\infty \sum_{k=1}^K \alpha_k w_{k,s,t} e^{-r(t-s)}\,\mathrm{d}t. \tag{OA.9}$$

This yields existence and uniqueness of the equilibrium, and proves Proposition 3.4.

We now proceed to the proof of Lemma B.3. Observe that, if Y is a linear rating that is stationary, then, up to an additive constant,

$$Y_t = \sum_k \int_{s \le t} u_k(t-s)\,\mathrm{d}S_{k,s}.$$

If $\mathbf{Var}[Y_t] = 1$, applying the projection formula for jointly Gaussian random variables, we get

$$\mu_t = \mathbf{E}^*[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[\theta_t, Y_t]}{\mathbf{Var}[Y_t]} (Y_t - \mathbf{E}^*[Y_t]) = \mathbf{Cov}[\theta_t, Y_t] \sum_k \int_{s \le t} u_k(t{-}t) \, (\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s) \, .$$

Thus,

$$w_{k,s,t} = \mathbf{Cov}[\theta_t, Y_t] u_k(t-t). \tag{OA.10}$$

We note that $\mathbf{Cov}[Y_t, \theta_t]$ is constant and equal to

$$\mathbf{Cov}[Y_t, \theta_t] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k(s) e^{-s} \,\mathrm{d}s.$$

Plugging (OA.10) into (OA.9) yields

$$c'(A_t) = \frac{\gamma^2}{2} \left[\sum_{k=1}^{K} \beta_k \int_0^\infty u_k(t) e^{-t} \,\mathrm{d}t \right] \left[\sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(t) e^{-rt} \,\mathrm{d}t \right].$$

OA.4 Complements to the Proof of Theorem 3.6 (and Th. B.4)

In this section, we prove Theorem B.4, which is the multisignal signal version of Theorem 3.6, and we show how to generate the educated guess used in that proof (which includes the guess candidate of Theorem 3.6 as a special case).

Derivation of the candidate guess

Assume that Y has the linear representation

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \,\mathrm{d}S_{k,s},$$

with u_k integrable and square integrable. It is helpful to make additional regularity assumptions to derive necessary conditions, so as to pin down a unique candidate for the coefficients u_k. Let $B > 0$, and assume that every u_k is twice continuously differentiable on $[0, B]$, and that $u_k(s) = 0$ if $s > B$. Below, we relax the bounded-support assumption. We define $U = \sum_k \beta_k u_k$.

We have

$$f_k(\tau) = \mathbf{Cov}[Y_t, S_{k,t-\tau}]$$

$$= \sum_{i=1}^{K} \int_0^\infty u_i(s)\,\mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\tau}]$$
$$= \sigma_k^2 \int_\tau^\infty u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2}\int_0^\infty \int_\tau^\infty U(s)e^{-|j-s|}\,\mathrm{d}j\,\mathrm{d}s.$$

Successive differentiations yield

$$f_k'(\tau) = -\sigma_k^2 u_k(\tau) - \frac{\beta_k\gamma^2}{2}\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s, \tag{OA.11}$$
$$f_k''(\tau) = -\sigma_k^2 u_k'(\tau) + \frac{\beta_k\gamma^2}{2}\int_0^\tau U(s)e^{-(\tau-s)}\,\mathrm{d}s - \frac{\beta_k\gamma^2}{2}\int_\tau^\infty U(s)e^{+(\tau-s)}\,\mathrm{d}s,$$
$$f_k'''(\tau) = -\sigma_k^2 u_k''(\tau) + \beta_k\gamma^2 U(\tau) - \frac{\beta_k\gamma^2}{2}\int_0^\tau U(s)e^{-(\tau-s)}\,\mathrm{d}s - \frac{\beta_k\gamma^2}{2}\int_\tau^\infty U(s)e^{+(\tau-s)}\,\mathrm{d}s.$$

Thus,

$$f_k' - f_k''' = \sigma_k^2 u'' - \sigma_k^2 u - \beta_k\gamma^2 U. \tag{OA.12}$$

Multiplying (OA.12) by β_k/σ_k^2 and summing over k yields an ordinary differential equation (ODE) for U:

$$\bar{f}' - \bar{f}''' = U'' - U - \gamma^2 m_\beta U = U'' - \kappa^2 U, \tag{OA.13}$$

where we recall that

$$\bar{f}(s) := \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} f_k(s).$$

Integrating by parts the general solution of (OA.13) gives

$$U(\tau) = C_1 e^{\kappa\tau} + C_2 e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2-1}{\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s, \tag{OA.14}$$

for some constants C_1 and C_2. Multiplying the expression for f_k' by β_k/σ_k^2 and summing over k gives

$$\bar{f}'(\tau) = -U(\tau) - \frac{\kappa^2-1}{2}\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s, \tag{OA.15}$$

for every $\tau \geq 0$. Together, and after simplification, (OA.14) and (OA.15) yield an equation that C_1 and C_2 should satisfy, for every τ:

$$\begin{aligned}\bar{f}'(\tau) = \bar{f}'(\tau) &- C_1 e^{\kappa\tau} - C_2 e^{-\kappa\tau} \\ &- \frac{\kappa^2-1}{2} C_1 \left[\frac{e^{B(\kappa-1)+\tau}}{\kappa-1} - \frac{e^{-\tau}}{\kappa+1} + \frac{e^{\kappa\tau}}{\kappa+1} - \frac{e^{\kappa\tau}}{\kappa-1} \right] \\ &- \frac{\kappa^2-1}{2} C_2 \left[-\frac{e^{-B(\kappa+1)+\tau}}{\kappa+1} + \frac{e^{-\tau}}{\kappa-1} + \frac{e^{-\kappa\tau}}{\kappa+1} - \frac{e^{-\kappa\tau}}{\kappa-1} \right] \\ &+ \frac{\kappa^2-1}{2} \frac{\kappa^2-1}{\kappa} \frac{e^{-B+\tau}}{\kappa^2-1} \int_0^B \bar{f}'(j) \left[\kappa \cosh(\kappa(B-j)) + \sinh(\kappa(B-j)) \right] \mathrm{d}j.\end{aligned}$$

After further simplification, we obtain a system of two equations in C_1 and C_2:

$$\begin{aligned}&-C_1 \frac{1}{\kappa-1} e^{B(\kappa-1)} + C_2 \frac{1}{\kappa+1} e^{-B(\kappa+1)} \\ &\qquad + \frac{e^{B(\kappa-1)}}{2\kappa} (\kappa+1) \int_0^B \bar{f}'(j) e^{-\kappa j} \,\mathrm{d}j + \frac{e^{-B(\kappa+1)}}{2\kappa} (\kappa-1) \int_0^B \bar{f}'(j) e^{\kappa j} \,\mathrm{d}j = 0,\end{aligned}$$

and

$$\frac{C_1}{\kappa+1} = \frac{C_2}{\kappa-1}.$$

Therefore, solving these two equations,

$$C_1 = \frac{1}{2\kappa} \frac{e^{B(\kappa-1)}(\kappa+1)^2(\kappa^2-1) \int_0^B \bar{f}'(j) e^{-\kappa j} \,\mathrm{d}j + e^{-B(\kappa+1)}(\kappa+1)^2(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{\kappa j} \,\mathrm{d}j}{(\kappa+1)^2 e^{B(\kappa-1)} - (\kappa-1)^2 e^{-B(\kappa+1)}},$$

and

$$C_2 = \frac{1}{2\kappa} \frac{e^{B(\kappa-1)}(\kappa+1)^2(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{-\kappa j} \,\mathrm{d}j + e^{-B(\kappa+1)}(\kappa^2-1)(\kappa-1)^2 \int_0^B \bar{f}'(j) e^{\kappa j} \,\mathrm{d}j}{(\kappa+1)^2 e^{B(\kappa-1)} - (\kappa-1)^2 e^{-B(\kappa+1)}}.$$

To get candidate coefficients whose support is not necessarily bounded, we send B to infinity and get

$$C_1 \to C_1^\infty := \frac{\kappa^2-1}{2\kappa} \int_0^\infty \bar{f}'(j) e^{-\kappa j} \,\mathrm{d}j, \tag{OA.16}$$

and

$$C_2 \to C_1^\infty := \frac{(\kappa-1)^2}{2\kappa} \int_0^\infty \bar{f}'(j) e^{-\kappa j} \,\mathrm{d}j. \tag{OA.17}$$

Thus, a candidate for U is

$$U(\tau) = C_1^\infty e^{\kappa\tau} + C_2^\infty e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2-1}{\kappa} \int_0^\tau \sinh(\kappa(\tau-s)) \bar{f}'(s) \,\mathrm{d}s.$$

We plug in the expression of U in (OA.11), which yields the candidate for u_k:

$$u_k(\tau) = C_1^\infty \frac{\beta_k \gamma^2}{\sigma_k^2(\kappa^2-1)} e^{\kappa\tau} + C_1^\infty \frac{\beta_k \gamma^2}{\sigma_k^2(\kappa^2-1)} e^{-\kappa\tau} - \frac{f_k'(\tau)}{\sigma_k^2} - \frac{\beta_k\gamma^2}{\sigma_k^2\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s, \tag{OA.18}$$

and after simplification,

$$u_k(\tau) = \frac{\beta_k\gamma^2}{\sigma_k^2\kappa}\left(\frac{\sinh\kappa\tau + \kappa\cosh\kappa\tau}{1+\kappa}\int_0^\infty e^{-\kappa s}\,\mathrm{d}\bar{f}(s) - \int_0^\tau \sinh\kappa(t-s)\,\mathrm{d}\bar{f}(s)\right) - \frac{f_k'(\tau)}{\sigma_k^2}. \tag{OA.19}$$

Proof that the candidate guess is correct

We show that the candidate for $\{u_k\}_k$ defines valid coefficients for the rating. Let u_k be defined by (OA.18), or, equivalently, by (OA.19). That u_k is integrable and square integrable was already demonstrated in the proof of Theorem 3.6 for the one-signal case, and continues to apply for the multisignal case.

Let

$$\widetilde{Y}_t := \mathbf{E}^*[Y_t] + \sum_{k=1}^K \int_{s\le t} u_k(t-s)(\mathrm{d}S_{k,s} - \alpha_k A_s^*\,\mathrm{d}s).$$

If we have $\mathbf{Cov}[Y_t - \widetilde{Y}_t, S_{k,t-\tau}] = 0$ for all τ and k, then Y_t and $S_{k,t-\tau}$ are independent for all τ and k. As $Y_t - \widetilde{Y}_t$ is measurable with respect to the information generated by the signals $S_{k,t-\tau}$, $\tau \ge 0$, $k = 1,\dots,K$, it implies that $\mathbf{Var}[Y_t - \widetilde{Y}_t] = 0$ and thus $Y_t = \widetilde{Y}_t$. In the remainder, we show that $\mathbf{Cov}[Y_t - \widetilde{Y}_t, S_{k,t-\tau}] = 0$ for all $\tau \ge 0$ and $k = 1,\dots,K$. Let $g_k(\tau) = \mathbf{Cov}[\widetilde{Y}_t, S_{k,t-\tau}]$. We have:

$$\begin{aligned} g_k(\tau) &= \sum_{i=1}^K \int_0^\infty u_i(s)\,\mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\tau}] \\ &= \sigma_k^2 \int_\tau^\infty u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2}\int_0^\infty\int_\tau^\infty U(s)e^{-|s-j|}\,\mathrm{d}j\,\mathrm{d}s, \end{aligned}$$

where $U(s) := \sum_i \beta_i u_i(s)$, and so

$$g_k'(\tau) = -\sigma_k^2 u_k(\tau) - \frac{\beta_k\gamma^2}{2}\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s.$$

Replacing u_k by its definition in (OA.18),

$$g_k'(\tau) = f_k'(\tau) - C_1\frac{\beta_k\gamma^2}{\kappa^2-1}e^{\kappa\tau} - C_2\frac{\beta_k\gamma^2}{\kappa^2-1}e^{-\kappa\tau} + \frac{\beta_k\gamma^2}{\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s - \frac{\beta_k\gamma^2}{2}\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s. \tag{OA.20}$$

Further, multiplying (OA.18) by β_k and summing over k, we have

$$U(\tau) = C_1^\infty e^{\kappa\tau} + C_2^\infty e^{-\kappa\tau} - \bar{f}'(\tau) - \frac{\kappa^2-1}{\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s.$$

It holds that

$$\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s = \lim_{B\to\infty}\int_0^B U(s)e^{-|\tau-s|}\,\mathrm{d}s.$$

Thus,

$$\int_0^B U(s)e^{-|\tau-s|}\,\mathrm{d}s = C_1^\infty\int_0^B e^{\kappa s}e^{-|\tau-s|}\,\mathrm{d}s + C_2^\infty\int_0^B e^{-\kappa s}e^{-|\tau-s|}\,\mathrm{d}s - \int_0^B \bar{f}'(s)e^{-|\tau-s|}\,\mathrm{d}s - \frac{\kappa^2-1}{\kappa}\int_0^B\int_0^s \sinh(\kappa(s-j))\bar{f}'(j)e^{-|\tau-s|}\,\mathrm{d}j\,\mathrm{d}s.$$

Then, for any $B > \tau$, we write

$$\int_0^B\int_0^s \sinh(\kappa(s-j))\bar{f}'(j)e^{-|\tau-s|}\,\mathrm{d}j\,\mathrm{d}s = -\frac{\kappa}{\kappa^2-1}\int_0^B \bar{f}'(j)e^{-|\tau-j|}\,\mathrm{d}j + \frac{e^{-B+\tau}}{\kappa^2-1}\int_0^B \bar{f}'(j)\left[\kappa\cosh(\kappa(B-j)) + \sinh(\kappa(B-j))\right]\mathrm{d}j - \frac{2}{\kappa^2-1}\int_0^\tau \sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j.$$

Using the expressions for C_1^∞ and C_2^∞ given by (OA.16) and (OA.17), we get that

$$C_1^\infty\left[\frac{e^{B(\kappa-1)+\tau}}{\kappa-1} - \frac{e^{-\tau}}{\kappa+1}\right] + C_2^\infty\left[-\frac{e^{-B(\kappa+1)+\tau}}{\kappa+1} + \frac{e^{-\tau}}{\kappa-1}\right] + \frac{\kappa^2-1}{\kappa}\frac{\kappa}{\kappa^2-1}\int_0^B \bar{f}'(j)e^{-|\tau-j|}\,\mathrm{d}j - \frac{\kappa^2-1}{\kappa}\frac{e^{-B+\tau}}{\kappa^2-1}\int_0^B \bar{f}'(j)\left[\kappa\cosh(\kappa(B-j)) + \sinh(\kappa(B-j))\right]\mathrm{d}j$$

converges to 0 as $B \to \infty$. Therefore,

$$\int_0^\infty U(s)e^{-|\tau-s|}\,\mathrm{d}s = \frac{2}{\kappa}\int_0^\tau \sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j + C_1^\infty\left[\frac{e^{\kappa\tau}}{\kappa+1}-\frac{e^{\kappa\tau}}{\kappa-1}\right] + C_2^\infty\left[\frac{e^{-\kappa\tau}}{\kappa+1}-\frac{e^{-\kappa\tau}}{\kappa-1}\right]. \tag{OA.21}$$

Plugging the expression of (OA.21) in (OA.20) yields

$$\begin{aligned} g_k'(\tau) = f_k'(\tau) &- C_1^\infty\frac{\beta_k\gamma^2}{\kappa^2-1}e^{\kappa\tau} - C_2^\infty\frac{\beta_k\gamma^2}{\kappa^2-1}e^{-\kappa\tau} + \frac{\beta_k\gamma^2}{\kappa}\int_0^\tau \sinh(\kappa(\tau-s))\bar{f}'(s)\,\mathrm{d}s \\ &- \frac{\beta_k\gamma^2}{\kappa}\int_0^\tau \sinh(\kappa(\tau-j))\bar{f}'(j)\,\mathrm{d}j - \frac{\beta_k\gamma^2}{2}C_1^\infty\left[\frac{e^{\kappa\tau}}{\kappa+1}-\frac{e^{\kappa\tau}}{\kappa-1}\right] \\ &- \frac{\beta_k\gamma^2}{2}C_2^\infty\left[\frac{e^{-\kappa\tau}}{\kappa+1}-\frac{e^{-\kappa\tau}}{\kappa-1}\right] = f_k'(\tau). \end{aligned}$$

So $g_k' = f_k'$. As $f_k(0) = g_k(0) = 0$, it follows that $f = g$. Uniqueness of the coefficients (up to measure zero sets) is immediate by linearity, as different coefficients on a set of positive measure yield a different joint distribution over ratings and signals.

Part III

Complements for Section 4

OA.5 Proof of Theorem 4.1 (and Th. 5.1 and B.8)

We prove Theorems 5.1 and B.8, which include as a special case Theorem 4.1. The constants m_α, m_β, $m_{\alpha\beta}$ and κ are defined in Section B.3.1 of Appendix B.

The first half of the proof derives a candidate optimal rating, obtained through first-order necessary conditions using a variational argument. The second half, self contained, verifies that the candidate rating just derived is optimal.

OA.5.1 Candidate Optimal Rating

We use the following shorthand notation:

$$U(t) \coloneqq \sum_{k=1}^{K} \beta_k u_k(t), \qquad V(t) \coloneqq \sum_{k=1}^{K} \alpha_k u_k(t),$$

$$U_0 \coloneqq \int_0^\infty U(t) e^{-t}\,\mathrm{d}t, \qquad V_0 \coloneqq \int_0^\infty V(t) e^{-rt}\,\mathrm{d}t.$$

We seek to maximize $c'(A)$ (where A is the stationary equilibrium action of the worker) among confidential information structures generated by rating processes with mean zero and with linear filter $\mathbf{u} \coloneqq \{u_k\}_k$, which in addition satisfy the normalization condition that the rating has variance one.

Any such rating process Y can be written as

$$Y_t = \sum_{k=1}^{K} \int_{s\le t} u_k(t-s)\,\mathrm{d}S_{k,s}.$$

We note that, by Itô's isometry,

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2\,\mathrm{d}s + \sum_{k=1}^{K}\sum_{k'=1}^{K} \int_{j\le t}\int_{i\le t} \beta_k\beta_{k'} u_k(t-i) u_{k'}(t-j)\,\mathbf{Cov}[\theta_i,\theta_j]\,\mathrm{d}i\,\mathrm{d}j,$$

and since θ is a stationary Ornstein-Uhlenbeck process with mean-reversion rate 1 and scale

γ, we have $\mathbf{Cov}[\theta_t, \theta_s] = \gamma^2 e^{-|t-s|}/2$, so that

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \, \mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j.$$

Applying Lemma B.3, the problem of maximizing $c'(A)$ among rating processes that satisfy the normalization condition thus reduces to choosing a linear filter $\mathbf{u}$ that maximizes

$$\left[\int_0^\infty V(t)e^{-rt} \, \mathrm{d}t\right] \left[\int_0^\infty U(t)e^{-t} \, \mathrm{d}t\right],$$

subject to

$$\frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t)^2 \, \mathrm{d}t = 1.$$

This optimization problem is a problem of calculus of variations with isoperimetric constraint. Assume there exists a solution $\mathbf{u}^*$ to this optimization problem, where $\mathbf{u}^*$ is twice differentiable, integrable, and square integrable.

Let

$$L(\mathbf{u}, \lambda_0) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u}),$$

where F and G are defined as

$$F(\mathbf{u}) = \left[\int_0^\infty V(t)e^{-rt} \, \mathrm{d}t\right] \left[\int_0^\infty U(t)e^{-t} \, \mathrm{d}t\right],$$

and

$$G(\mathbf{u}) = \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t)^2 \, \mathrm{d}t.$$

The function L defines an unconstrained maximization problem for every given λ_0. It corresponds to the Lagrangian of the constrained optimization problem up to an additive, $\mathbf{u}$-independent term, where the coefficient λ_0 is a Lagrangian multiplier. However, we do not need to invoke the Theorem of Lagrange Multipliers and its extensions to isoperimetric problems in the calculus of variations. Instead, we will look for a constant λ_0 that yields a unique candidate of the unconstrained maximization problem that satisfies the Euler-Lagrange first-order conditions, and, in addition, satisfies the original constraint. In the remainder of this proof, we refer to the unconstrained optimization problem as the *relaxed* optimization problem, as opposed to the *original* (constrained) maximization problem.

Observe that we can write both F and G as a double integral, namely,

$$F(\mathbf{u}) = \int_0^\infty \int_0^\infty V(i)U(j)e^{-ri}e^{-j} \, \mathrm{d}i \, \mathrm{d}j,$$

while

$$G(\mathbf{u}) = \int_0^\infty \int_0^\infty \left(\frac{\gamma^2}{2} U(i)U(j)e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^K \sigma_k^2 u_k(j)^2 e^{-i} \right) \mathrm{d}i\,\mathrm{d}j.$$

This enables the application of Proposition OA.1 in Part VI of this Online Appendix. Assume there exists a $\lambda_0^* < 0$ such that $\mathbf{u} = \mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda_0^*)$.[1]

Proposition OA.1 gives the first-order condition derived from the Euler-Lagrange equations: if $\lambda_0 = \lambda_0^*$ and $\mathbf{u} = \mathbf{u}^*$, then for all k and all t, we have $L_k(t) = 0$, where

$$L_k(t) := \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t} + \lambda_0 \gamma^2 \beta_k \int_0^\infty U(j) e^{-|t-j|}\,\mathrm{d}j + 2\lambda_0 \sigma_k^2 u_k(t) = 0, \quad \text{(OA.22)}$$

and where U_0, V_0, U and V are defined as above as an implicit function of $\mathbf{u}$.

We differentiate the above equation in the variable t twice, and get, for all k and all t:

$$\alpha_k U_0 r^2 e^{-rt} + \beta_k V_0 e^{-t} - 2\lambda_0\gamma^2\beta_k U(t) + \lambda_0\gamma^2\beta_k \int_0^\infty U(j)e^{-|t-j|}\,\mathrm{d}j + 2\lambda_0\sigma_k^2 u_k''(t) = 0. \quad \text{(OA.23)}$$

The difference between (OA.22) and (OA.23) is

$$(1-r^2)\alpha_k U_0 e^{-rt} + 2\lambda_0\gamma^2\beta_k U(t) + 2\lambda_0\sigma_k^2(u_k(t) - u_k''(t)) = 0. \quad \text{(OA.24)}$$

In particular, multiplying (OA.24) by β_k/σ_k^2 and summing over k, we get a linear differential equation that $U(t)$ must satisfy, namely,

$$(1-r^2)m_{\alpha\beta}U_0 e^{-rt} + 2\lambda_0\gamma^2 m_\beta U(t) + 2\lambda_0(U(t) - U''(t)) = 0,$$

where we recall that $m_\beta = \sum_k \beta_k^2/\sigma_k^2$, $m_{\alpha\beta} = \sum_k \alpha_k\beta_k/\sigma_k^2$, and $m_\alpha = \sum_k \alpha_k^2/\sigma_k^2$.

The characteristic polynomial has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$. A particular solution is Ce^{-rt}, for some constant C. We have assumed integrable functions u_k, so in the general solution we retain the two exponentials with negative rates, and get

$$U(t) = C_1 e^{-rt} + C_2 e^{-\kappa t},$$

for some constants C_1 and C_2.

For such U, u_k satisfies the linear differential equation (OA.24), whose characteristic polynomial has roots ± 1. A particular solution is a sum of scaled time exponentials e^{-rt}

[1]Insofar as we find a coefficient λ_0^* that yields a unique candidate which is shown to solve the original problem, we need not prove uniqueness of the coefficient. However, it is easily seen that $\lambda_0^* < 0$ is a necessary second-order condition. The optimum marginal cost, if it exists, is strictly positive, *i.e.*, $F(\mathbf{u}^*) > 0$, since the optimal solution does at least as well as transparency (giving all information included in all signals to the market) and transparency induces a positive equilibrium effort by our assumption that $m_{\alpha\beta} > 0$. This implies $\lambda_0^* < 0$, because $F(\mathbf{u}^*) > 0$ and $G(\mathbf{u}^*) = 1$.

and $e^{-\kappa t}$. As every u_k is bounded, we must consider the negative root of the characteristic equation, and we get that

$$u_k(t) = D_{1,k}e^{-rt} + D_{2,k}e^{-\kappa t} + D_{3,k}e^{-t}, \tag{OA.25}$$

for some constants $D_{1,k}, D_{2,k}, D_{3,k}$.

Determination of the Constants

We have established that the solution belongs to the family of functions that are sums of scaled time exponentials. We now solve for the constant factors.

We plug in the general form of u_k from (OA.25) in the expression for L_k, and get:

$$L_k = L_{1,k}e^{-rt} + L_{2,k}e^{-\kappa t} + L_{3,k}e^{-t},$$

where the coefficients $L_{1,k}$, $L_{2,k}$, $L_{3,k}$ depend on the primitives of the model and the constants $D_{1,k}, D_{2,k}, D_{3,k}$. The condition that $L_k = 0$ implies that $L_{1,k} = L_{2,k} = L_{3,k} = 0$.

First, note that $U(t)$ does not include a term of the form e^{-t}, which implies that

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0. \tag{OA.26}$$

We also observe that

$$L_{2,k} = 2\lambda_0\sigma_k^2 D_{2,k} - \frac{2\gamma^2\lambda_0\beta_k \sum_{i=1}^{K} \beta_i D_{2,i}}{\kappa^2 - 1},$$

so that $L_{2,k} = 0$ for all k implies

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}, \tag{OA.27}$$

for some multiplier a. Next, we use (OA.27) together with (OA.26) to show that

$$\begin{aligned} L_{3,k} = {} & \frac{\beta_k}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{\beta_k}{r+1}\sum_{i=1}^{K}\alpha_i D_{3,i} + \frac{\gamma^2\lambda_0\beta_k}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} + 2\lambda_0\sigma_k^2 D_{3,k} \\ & + \frac{a\gamma^2\lambda_0\beta_k m_\beta}{\kappa - 1} + \frac{a\beta_k m_{\alpha\beta}}{\kappa + r}, \end{aligned}$$

and $L_{3,k} = 0$ for every k implies that $D_{3,k} = 0$ for all k. The equation $L_{3,k}/\beta_k = 0$ is linear in λ_0, and then simplifies to:

$$\lambda_0\left(\frac{\gamma^2}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} + \frac{a\gamma^2 m_\beta}{\kappa - 1}\right) + \frac{1}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{a m_{\alpha\beta}}{\kappa + r} = 0. \tag{OA.28}$$

Next, we use (OA.27) together with (OA.26) to show that

$$L_{1,k} = 2\lambda_0\sigma_k^2 D_{1,k} + \frac{a\alpha_k m_\beta}{\kappa+1} + \frac{\left((r-1)\alpha_k - 2\gamma^2\lambda_0\beta_k\right)}{r^2-1}\sum_{i=1}^{K}\beta_i D_{1,i},$$

and, since $L_{1,k}=0$ must hold for every k, we get, since $\lambda_0 \neq 0$,

$$\sigma_k^2 D_{1,k} = \left(\frac{\gamma^2\beta_k}{r^2-1} - \frac{\alpha_k}{2\lambda_0 + 2\lambda_0 r}\right)\sum_{i=1}^{K}\beta_i D_{1,i} - \frac{a\alpha_k m_\beta}{2\kappa\lambda_0 + 2\lambda_0}. \tag{OA.29}$$

We multiply (OA.29) by β_k/σ_k^2, and sum over k to get

$$\left[(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right)\right]\sum_{i=1}^{K}\beta_i D_{1,i} = -a\left(r^2-1\right)m_{\alpha\beta}m_\beta.$$

As by assumption $r \neq 1$, the right-hand side is nonzero, which implies

$$\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right) \neq 0, \tag{OA.30}$$

and thus

$$\sum_{i=1}^{K}\beta_i D_{1,i} = \frac{-a\left(r^2-1\right)m_{\alpha\beta}m_\beta}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right)}. \tag{OA.31}$$

Similarly, if we multiply (OA.29) by α_k/σ_k^2 and sum over k, we get

$$\begin{aligned}\sum_{i=1}^{K}\alpha_i D_{1,i} &= \left(\frac{\gamma^2 m_{\alpha\beta}}{r^2-1} - \frac{m_\alpha}{2\lambda_0 + 2\lambda_0 r}\right)\sum_{i=1}^{K}\beta_i D_{1,i} - \frac{a m_\alpha m_\beta}{2\kappa\lambda_0 + 2\lambda_0} \\ &= \frac{a m_\beta\left(m_\alpha\left(\gamma^2 m_\beta - r^2 + 1\right) - \gamma^2 m_{\alpha\beta}^2\right)}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda_0(r+1)\right) - 2\gamma^2\lambda_0 m_\beta\right)}.\end{aligned}$$

Putting together (OA.28), (OA.31) and (OA.5.1) yield a quadratic equation in λ_0 of the form

$$A\lambda_0^2 + B\lambda_0 + C = 0, \tag{OA.32}$$

with coefficients given by, after simplifications using in particular that $\kappa^2 = 1 + \gamma^2 m_\beta$,

$$A = m_\beta \frac{\kappa + r}{1-\kappa},$$

$$\begin{aligned}
B &= \frac{m_{\alpha\beta}\left(\gamma^2 m_\beta\left(-2\kappa^2+r^2+1\right)+\left(\kappa^2-1\right)\left(r^2-1\right)\right)}{\gamma^2\left(\kappa^2-1\right)\left(\gamma^2 m_\beta - r^2+1\right)} \\
&= -\frac{2}{\gamma^2} m_{\alpha\beta}, \\
C &= \frac{m_\alpha m_\beta(\kappa+r)\left(r^2-\kappa^2\right)+m_{\alpha\beta}^2\left(\gamma^2 m_\beta(\kappa+r)-2(\kappa+1)(r-1)r\right)}{4\gamma^2(\kappa+1)r\left(r^2-\kappa^2\right)} \\
&= \frac{(\kappa-1)m_\alpha(\kappa+r)^2-\gamma^2 m_{\alpha\beta}^2(\kappa+2r-1)}{4\gamma^4 r(\kappa+r)}.
\end{aligned}$$

As $\kappa > 1$, we immediately have $A < 0$. Also, C has the sign of

$$(\kappa-1)m_\alpha(\kappa+r)^2 - m_{\alpha\beta}^2(\kappa-1+2r)\gamma^2 = (\kappa-1)m_\alpha(\kappa+r)^2 - m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1).$$

By the Cauchy-Schwarz inequality, $m_\alpha m_\beta \geq m_{\alpha\beta}^2$, so

$$\begin{aligned}
(\kappa-1)m_\alpha(\kappa+r)^2 - m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1) &\geq m_\alpha\left\{(\kappa-1)(\kappa+r)^2-(\kappa-1+2r)(\kappa^2-1)\right\} \\
&= m_\alpha(\kappa-1)(1-r)^2 > 0.
\end{aligned}$$

Hence C is positive, $A \cdot C$ is negative, and Equation (OA.32) has two roots, one positive and one negative. Besides, as $m_{\alpha\beta} > 0$ by assumption, $B < 0$. As we have already established that λ_0 must be negative, we conclude that

$$\lambda_0 = \frac{-B+\sqrt{B^2-4AC}}{2A}.$$

Pulling out the term $\sum_i \beta_i D_{1,i}$ in (OA.29) using (OA.31), we express $D_{1,k}$ as a solution of the linear equation. It follows that

$$D_{1,k} = a\frac{m_\beta\left[\gamma^2 m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2}-(\kappa^2-r^2)\frac{\alpha_k}{\sigma_k^2}\right]}{(1+\kappa)\left[2\lambda_0(\kappa^2-r^2)+(1-r)m_{\alpha\beta}\right]},$$

where the denominator is nonzero by (OA.30). We can simplify those expressions further. We define

$$\lambda = (\kappa-1)\sqrt{r}(1+r)m_{\alpha\beta}+(\kappa-r)\sqrt{\Delta},$$

where

$$\Delta \coloneqq (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2)+(1+r)^2 m_{\alpha\beta}^2.$$

Then, $D_{1,k} = a\sqrt{r}c_k/\lambda$ with

$$c_k \coloneqq (\kappa^2-r^2)m_\beta\frac{\alpha_k}{\sigma_k^2}+(1-\kappa^2)m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2}.$$

Note that, as a rating process induces the same effort level up to a scaling of the rating process, any multiplier a yields the same equilibrium action. Thus, a candidate optimal rating process for the original optimization problem is given by the linear filter

$$u_k(t) = c_k \frac{\sqrt{r}}{\lambda} e^{-rt} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t}, \ \forall k.$$

If

$$a = \frac{(\kappa - 1)\left((\kappa - 1) m_{\alpha\beta}(r+1)\sqrt{r} + \sqrt{\Delta}(\kappa - r)\right)}{2\sqrt{\Delta} m_\beta (\kappa - r)},$$

then the conditions of Proposition B.1 are satisfied, so that the associated rating process is a market belief for a confidential information structure.

OA.5.2 Verification of Optimality

In this (sub)section, we establish that the candidate rating obtained above is indeed optimal. We consider an auxiliary principal-agent setting. We refer to the principal-agent setting as the auxiliary setting, and to the main setting detailed in the main text as the original setting.

Auxiliary Setting. In the auxiliary setting, there is a principal (a female) and an agent (a male). Time $t \geq 0$ is continuous and the horizon infinite. The agent is as the worker in the original model. He exerts private effort (his action), has an exogenous random ability, produces output X and generates signals $S_1 = X, S_2, \ldots, S_K$ over time. The various laws of motion, for the agent's ability, output, signals, are as in those of the worker in the original setting. The filtration $\mathcal{R}$ captures all information of the signal processes, as defined in Section 3 and Appendix A. The agent's information at time t is also $\mathcal{R}_t$. The agent's strategy, which specifies his private action at every moment as a function of his information, continues to be a bounded $\mathcal{R}$-adapted process A.

However, the agent's payoff is not the same as the worker's payoff of the original model. In the auxiliary setting, the agent is not paid by a market, but by a principal. Informally, over the interval $[t, t + \mathrm{d}t)$, the principal transfers the amount $Y_t \, \mathrm{d}t$ to the agent. Here, Y is a stochastic process interpreted as a transfer rate (payments may be negative). The agent is risk-neutral; he discounts future payoffs at rate $r > 0$, and his instantaneous cost of effort is $c(\cdot)$, as in the original setting. The agent's realized discounted payoff is

$$\int_0^\infty e^{-rt} \left(Y_t - c(A_t)\right) \mathrm{d}t.$$

Given Y, the agent chooses a strategy A that maximizes his expected discounted payoff, namely,

$$A \in \underset{\widehat{A}}{\operatorname{argmax}} \mathbf{E}\left[\int_0^\infty e^{-rt}\left(Y_t - c(\widehat{A}_t)\right) \mathrm{d}t \,\middle|\, \mathcal{R}_0\right], \tag{OA.33}$$

where the expectation is under the law of motion defined by strategy $\widehat{A}$. A strategy that satisfies (OA.33) is called a *best response* to the transfer process Y.

In the auxiliary setting, the principal combines features of both the market and the rater in the original setting. As the market, the principal sets the transfer to the agent, and as the rater, she observes all the signals the agent generates over time, *i.e.*, she knows $\mathcal{R}_t$ at time t. The principal recommends a strategy to the agent, denoted A^*—the analogue of the market conjecture in the original setting. She is risk-neutral and has discount rate $\rho \in (0, r)$. Her payoff is

$$\int_0^\infty e^{-\rho t} H_t \, \mathrm{d}t.$$

For now, there is no need to specify the instantaneous payoff process H. We specialize H below as we discuss the principal's optimization program.

A *contract* for the principal is a pair (A^*, Y). The contract is *incentive compatible* if A^* is a best response to Y. For the most part, we focus on *stationary linear contracts.* These are contracts whose transfer processes Y are affine in the past signal increments, and are stationary: that is, there exist u_k, $k = 1, \dots, K$, such that, up to an additive constant,

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \, \mathrm{d}S_{k,s}.$$

The principal wants to maximize her own payoff over all contracts that are incentive compatible. This implies that there are two optimal control problems, one embedded into the other. First, we solve the agent's problem, and then turn to the principal's problem.

The Agent's Problem

We first state conditions of incentive compatibility. The proof follows the same arguments as in Lemma B.3.

Lemma OA.1 *Let (A, Y) be a stationary linear contract. The contract is incentive compatible if, and only if,*

$$c'(A) = \sum_{k=1}^{K} \alpha_k \int_0^\infty u(t) e^{-rt} \, \mathrm{d}t.$$

As is common in principal-agent problems, to solve the principal's problem using a dynamic programming approach, we express incentive compatibility in terms of the evolution of the agent's continuation value, or equivalently, the agent's continuation transfer. In the sequel, as in the main body of the paper, $\nu_t = \mathbf{E}[\theta_t \mid \mathcal{R}_t]$ is the agent's best current estimate about his ability.

Lemma OA.2 *Let (A, Y) be a stationary linear contract. If the contract is incentive compatible, then there exists constants $C_1, \ldots, C_K$, such that the agent's continuation transfer process J defined by*

$$J_t = \mathbf{E}\left[\int_{s\geq t} e^{-r(s-t)} Y_s \,\mathrm{d}s \;\middle|\; \mathcal{R}_t\right],$$

(where the expectation is taken with respect to the law induced by strategy A) satisfies the SDE

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + C_k\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\,\mathrm{d}t\right],$$

and the two transversality conditions

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{R}_t] = 0, \text{ and}$$
$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau}^2 \mid \mathcal{R}_t] = 0,$$

where $\xi_\beta := \sum_{k=1}^{K} \beta_k C_k$. In addition, the equilibrium action is defined by $c'(A_t) = \xi_\alpha := \sum_{k=1}^{K} \alpha_k C_k$.

Note that transversality is with respect to the principal's discount rate, not the agent's.
Proof. Consider a stationary linear contract (A, Y), where

$$Y_t = \sum_{k=1}^{K}\int_{s\leq t} u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

Let $J_T := \mathbf{E}\left[\int_{t\geq T} e^{-r(t-T)} Y_t\,\mathrm{d}t \;\middle|\; \mathcal{R}_T\right]$. We compute

$$\int_{t\geq T} e^{-r(t-T)} Y_t\,\mathrm{d}t = \sum_{k=1}^{K}\int_{t\geq T}\int_{s\leq T} e^{-r(t-T)} u_k(t-s)\left[\mathrm{d}S_{k,s} - A_s\,\mathrm{d}s\right]\mathrm{d}t$$
$$+ \sum_{k=1}^{K}\int_{s\geq T}\int_{t\geq s} e^{-r(t-T)} u_k(t-s)\,\mathrm{d}t\left[\mathrm{d}S_{k,s} - A_s\,\mathrm{d}s\right].$$

Note that, for $t \geq T$, $\mathbf{E}[\theta_t \mid \mathcal{R}_T, \theta_T] = \mathbf{E}[\theta_t \mid \theta_T] = e^{-(t-T)}\theta_T$, so using the law of iterated expectations, $\mathbf{E}[\theta_t \mid \mathcal{R}_T] = \mathbf{E}[\mathbf{E}[\theta_t \mid \mathcal{R}_T, \theta_T] \mid \mathcal{R}_T] = \mathbf{E}[e^{-(t-T)}\theta_T \mid \mathcal{R}_T] = e^{-(t-T)}\nu_T$. Hence, we can compute J_T as

$$\begin{aligned} J_T &= \sum_{k=1}^{K} \int_{t\geq T} \int_{s\leq T} e^{-r(t-T)} u_k(t-s)\, [\mathrm{d}S_{k,s} - A_s\, \mathrm{d}s]\, \mathrm{d}t \\ &\quad + \sum_{k=1}^{K} \beta_k \int_{s\geq T} \int_{t\geq s} e^{-r(t-T)} u_k(t-s) e^{-(s-T)} \nu_T\, \mathrm{d}t \\ &= \int_{t\geq T} \int_{s\leq T} e^{-r(t-T)} u(t-s)\, [\mathrm{d}S_{k,s} - A_s\, \mathrm{d}s]\, \mathrm{d}t + \frac{\nu_T}{1+r} \sum_{k=1}^{K} \beta_k \int_{\tau\geq 0} e^{-r\tau} u_k(\tau)\, \mathrm{d}\tau. \end{aligned}$$

Now, let us define the constants $C_1, \dots, C_K$ as

$$C_k = \int_{\tau\geq 0} e^{-r\tau} u_k(\tau)\, \mathrm{d}\tau.$$

Then

$$\begin{aligned} \mathrm{d}J_T &= \frac{\xi_\beta}{1+r}\, \mathrm{d}\nu_T - Y_T\, \mathrm{d}T + \sum_{k=1}^{K} C_k\, [\mathrm{d}S_{k,T} - \alpha_k A_T\, \mathrm{d}T] + rJ_T\, \mathrm{d}T - \frac{r}{1+r} \xi_\beta \nu_T\, \mathrm{d}T \\ &= \frac{\xi_\beta}{1+r}\, \mathrm{d}\nu_T + (rJ_T - Y_T)\, \mathrm{d}T + \sum_{k=1}^{K} C_k \left[\mathrm{d}S_{k,T} - \left(\alpha_k A_T + \frac{r}{1+r} \beta_k \nu_t \right) \mathrm{d}T \right]. \end{aligned}$$

After simplification, and using that $\mathrm{d}\nu_t = -\kappa\nu_t\, \mathrm{d}t + \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} [\mathrm{d}S_{k,t} - \alpha_k A_t\, \mathrm{d}t]$, we get

$$\mathrm{d}J_t = (rJ_t - Y_t)\, \mathrm{d}t + \sum_{k=1}^{K} \left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + C_k \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\, \mathrm{d}t]\, .$$

That $c'(A_t) = \xi_\alpha$ follows from Lemma OA.1. ■

Lemma OA.3 *Let (A, Y) be a stationary linear contract. Suppose J and $\widehat{C}_1, \dots, \widehat{C}_K$ are $\mathcal{R}$-adapted processes, and that J satisfies the SDE*

$$\mathrm{d}J_t = (rJ_t - Y_t)\, \mathrm{d}t + \sum_{k=1}^{K} \left(\widehat{\xi}_{\beta,t} \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\, \mathrm{d}t]\, , \tag{OA.34}$$

and the two transversality conditions

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{R}_t] = 0, \text{ and} \tag{OA.35}$$

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau}^2 \mid \mathcal{R}_t] = 0, \tag{OA.36}$$

where $\widehat{\xi}_\beta := \sum_k \beta_k \widehat{C}_k$.

Then, J_t *is the agent's continuation transfer* $\mathbf{E}\left[\int_{s\geq t} e^{-r(s-t)} Y_s \, \mathrm{d}s \,\middle|\, \mathcal{R}_t\right]$*, the contract is incentive compatible, and the agent's equilibrium action satisfies* $c'(A_t) = \sum_k \alpha_k \widehat{C}_k$.

Proof. We fix a stationary linear contract (A, Y). Let J and $\widehat{C}_1, \ldots, \widehat{C}_K$ be $\mathcal{R}$-adapted processes such that J satisfies (OA.34), subject to (OA.35) and (OA.36).

Integrating J yields

$$J_t - e^{-r\tau} J_{t+\tau} = \int_t^{t+\tau} e^{-r(s-t)} \left[Y_s - \sum_{k=1}^K \left(\widehat{\xi}_{\beta,t} \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\, \mathrm{d}t] \right],$$

and using that J is $\mathcal{R}$-adapted, together with the law of iterated expectations, we get

$$\begin{aligned} &J_t - \mathbf{E}\left[e^{-r\tau} J_{t+\tau} \,\middle|\, \mathcal{R}_t\right] \\ &= \mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)} Y_s \,\middle|\, \mathcal{R}_t\right] \\ &\quad + \sum_{k=1}^K \mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)} \left(\widehat{\xi}_{\beta,t} \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\, \mathrm{d}t] \,\middle|\, \mathcal{R}_t\right] \\ &= \mathbf{E}\left[\int_t^{t+\tau} e^{-r(s-t)} Y_s \,\middle|\, \mathcal{R}_t\right]. \end{aligned}$$

Taking the limit as $\tau \to +\infty$ and applying the transversality condition (OA.35), we get $J = V$, where V is the agent's continuation transfer, namely,

$$V_t := \mathbf{E}\left[\int_t^{\infty} e^{-r(s-t)} Y_s \,\middle|\, \mathcal{R}_t\right].$$

As in the proof of Lemma OA.2, for any stationary linear contract—incentive compatible or not—and an arbitrary strategy A of the agent, we have that

$$\mathrm{d}V_t = [rV_t - Y_t]\, \mathrm{d}t + \sum_{k=1}^K \left(\xi_\beta \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + C_k \right) [\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t)\, \mathrm{d}t],$$

with $C_k := \int_{\tau\geq 0} e^{-r\tau} u_k(\tau)\, \mathrm{d}\tau$. That $J = V$ implies $\widehat{C}_k = C_k$, and thus, by Lemma OA.1, the contract is incentive compatible. ∎

The Principal's Problem

The problem for the principal is to choose a contract (A, Y) such that two conditions are satisfied:

1. The process Y maximizes

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t} H_t \,\mathrm{d}t \,\middle|\, \mathcal{R}_0\right].$$

2. The contract is incentive compatible.

In the remainder of this proof, as instantaneous payoff for the principal, we use

$$H_t := c'(A_t) - \phi Y_t(Y_t - \nu_t), \tag{OA.37}$$

where

$$\phi := \frac{\sqrt{\Delta}}{\sqrt{r}(\kappa - 1)(r + \kappa)} > 0, \tag{OA.38}$$

and $\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$, as defined in Section B.3.1.

Remarks on the choice of the principal's payoff. In the original setting, the rater seeks to maximize the agent's discounted output. In a stationary setting, it is equivalent to maximizing the agent's discounted marginal cost. The marginal cost is the first term in the right-hand side of (OA.37). However, in the original setting, the agent's incentives are driven by the market's belief process. By Proposition B.1, the market belief process μ satisfies

$$\mu_t = \mathbf{E}[\theta_t \mid \mu_t] = \mathbf{E}[\nu_t \mid \mu_t] = \frac{\mathbf{Cov}[\mu_t, \nu_t]}{\mathbf{Var}[\mu_t]}\mu_t,$$

using the law of iterated expectations and the projection formula for jointly normal random variables. Thus $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$. To make the principal's payoff in the auxiliary setting and the rater's objective of the original setting comparable, we include a penalty term $\phi\mu_t(\nu_t - \mu_t)$ in the principal's payoff. Note that $\mathbf{E}\left[Y_t(\nu_t - Y_t)\right] = \mathbf{Cov}[Y_t, \nu_t] - \mathbf{Var}[Y_t]$. As a Lagrangian multiplier, the parameter ϕ captures the tradeoff between the maximization of the agent's marginal cost and the penalty term, so as to constrain the transfer to be close to a market belief. Its specific value (given in (OA.38)) is picked using the candidate optimal rating derived in the first half of this proof.

The principal's problem is an optimal control problem with two natural state variables: the agent's estimate of his ability, ν, and the agent's continuation transfer J. The state ν appears explicitly in the principal's payoff. Recall that ν can be expressed in closed form, namely,

$$\nu_t = \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \int_{s \le t} e^{-\kappa(t-s)} \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right].$$

Thus, for $t \ge 0$, the state variable ν is determined by its initial value,

$$\nu_0 = \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \int_{s \le 0} e^{\kappa s} \, \mathrm{d}S_{k,s},$$

and the equation of evolution of ν, namely,

$$\mathrm{d}\nu_t = -\kappa \nu_t \, \mathrm{d}t + \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right].$$

The other state J does not appear explicitly in the principal's payoff, but must be controlled to ensure that the transversality conditions are satisfied—by Lemmas OA.2 and OA.3, these transversality conditions are necessary and sufficient to ensure that the contract is incentive compatible.

The principal's problem can then be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$, so as to maximize, for all t,

$$\mathbf{E} \left[\int_t^{\infty} \rho e^{-\rho(s-t)} (c'(A_t) - \phi Y_t (Y_t - \nu_t)) \, \mathrm{d}s \;\middle|\; \mathcal{R}_t \right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha$, where we recall that $\widehat{\xi}_\alpha = \sum_k \alpha_k \widehat{C}_k$.

2. The evolution of the agent's belief ν, given by

$$\mathrm{d}\nu_t = -\kappa \nu_t \, \mathrm{d}t + \frac{\gamma^2}{1+\kappa} \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right].$$

3. The evolution of the agent's continuation transfer J, given by

$$\mathrm{d}J_t = (rJ_t - Y_t) \, \mathrm{d}t + \sum_{k=1}^{K} \left(\widehat{\xi}_{\beta,t} \frac{\gamma^2}{(1+\kappa)(1+r)} \frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \right) \left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k \nu_t) \, \mathrm{d}t \right],$$

where we recall that $\widehat{\xi}_\beta = \sum_k \beta_k \widehat{C}_k$.

4. The transversality conditions, given by

$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J_{t+\tau} \mid \mathcal{R}_t] = 0, \text{ and}$$
$$\lim_{\tau\to+\infty} \mathbf{E}[e^{-\rho\tau} J^2_{t+\tau} \mid \mathcal{R}_t] = 0.$$

To solve the principal's problem, we use dynamic programming. The principal maximizes

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)}(\xi_{\alpha,t} - \phi Y_t(Y_t - \nu_t))\,\mathrm{d}s \,\middle|\, \mathcal{R}_t\right],$$

for every t, subject to the evolution of the state variables ν and J, and the transversality conditions on J. Without the restriction to stationary linear transfer processes, the dynamic programming problem is standard. We solve the principal's problem without imposing that restriction, and verify *ex post* that the optimal transfer in this relaxed problem is indeed stationary linear.

Assume the principal's value function V is $\mathcal{C}^2(\mathbf{R}^2)$, as a function of the two states J and ν. By standard arguments, an application of Itô's Lemma yields the Hamilton-Jacobi-Bellman (HJB) equation for V, namely

$$\begin{aligned}\rho V = \sup_{y,c_1,\ldots,c_K} & \rho\widehat{\xi}_\alpha - \rho\phi y(y-\nu) + (rJ-y)V_J - \nu_t V_\nu + \gamma^2\frac{\kappa-1}{\kappa+1}V_{\nu\nu} \\ & + \frac{(\kappa+r)\gamma^2\widehat{\xi}_\beta}{(1+\kappa)(1+r)}V_{\nu J} + \sum_k\left(\widehat{\xi}_\beta\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k} + \sigma_k c_k\right)^2 V_{JJ},\end{aligned} \tag{OA.39}$$

where to shorten notation we have used the subscript notation for the (partial) derivatives of V, and have abused notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$V(J,\nu) = a_0 + a_1 J + a_2\nu + a_3 J\nu + a_4 J^2 + a_5\nu^2. \tag{OA.40}$$

Using the general form of the conjectured value function (OA.40), we can solve for $y, c_1, \ldots, c_K$ using the first-order condition. We can then plug these variables expressed as a function of the coefficients a_i's back into (OA.39), which allows to uniquely identify the coefficients. We obtain

$$a_0 = -\frac{m^2_{\alpha\beta}(\kappa-1)(1+2r+\kappa)}{4m_\beta(\kappa+r)^2(2r-\rho)\phi} + \frac{m_{\alpha\beta}(\kappa-1)}{2m_\beta(\kappa+r)} + \frac{(\kappa-1)^2\phi}{2m_\beta(2+\rho)} + \frac{m_\alpha}{4(2r-\rho)\phi},$$
$$a_1 = 0, a_2 = 0, a_3 = \frac{(2r-\rho)\rho\phi}{1+r}, a_4 = -(2r-\rho)\rho\phi, \text{ and } a_5 = \frac{\rho(1-r+\rho)^2\phi}{4(1+r)^2(2+\rho)}.$$

It is readily verified that the second-order condition is equivalent to $a_5 < 0$, and so it is satisfied for all $\rho < r$. After simplification, we obtain the following expressions for y and c_k:

$$y(J,\nu) = (2r-\rho)J + \frac{1-r+\rho}{2(1+r)}\nu, \text{ and} \tag{OA.41}$$

$$c_k(J,\nu) = \frac{\alpha_k}{2\sigma_k^2(2r-\rho)\phi} - \frac{\beta_k(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi\right)}{2\sigma_k^2 m_\beta (r+\kappa)^2(2r-\rho)\phi}.$$

Thus, we obtain that the optimal processes $\widehat{C}_k$ are constant, and we get the optimal transfer at time t, Y_t, as a linear function of the state variables J_t, ν_t:

$$Y_t = \begin{bmatrix} 2r-\rho \\ \frac{1-r+\rho}{2(1+r)} \end{bmatrix} \cdot \begin{bmatrix} J_t \\ \nu_t \end{bmatrix}. \tag{OA.42}$$

We insert the optimal control Y_t back into the equations that determine the evolution of the state variables. Doing so yields a linear two-dimensional stochastic differential equation, namely

$$\mathrm{d}\begin{bmatrix} J_t \\ \nu_t \end{bmatrix} = M \begin{bmatrix} J_t \\ \nu_t \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \widehat{\xi}_{\beta,t} \dfrac{\gamma^2}{(1+\kappa)(1+r)} \dfrac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \\ \dfrac{\kappa-1}{m_\beta} \dfrac{\beta_k}{\sigma_k^2} \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where

$$M := \begin{bmatrix} -r+\rho & -\widehat{\xi}_\beta \dfrac{\kappa+r}{1+r} - \dfrac{1-r+\rho}{2(1+r)} \\ 0 & -\kappa \end{bmatrix}.$$

The matrix M has two eigenvalues, $-(r-\rho)$ and $-\kappa$, which are generically distinct, and negative for $\rho < r$. We can write

$$\begin{bmatrix} J_t \\ \nu_t \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k e^{-(r-\rho)(t-s)} + \mathbf{g}_k e^{-\kappa(t-s)}\right) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where $\mathbf{f}_k$ and $\mathbf{g}_k$ are two-dimensional vectors that can be expressed in closed form as:

$$\mathbf{f}_k := \begin{bmatrix} \frac{m_\beta(r+\kappa)(r-\kappa-\rho)}{2m_\beta(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi} \frac{\alpha_k}{\sigma_k^2} + \frac{m_{\alpha\beta}(\kappa-1)(1+\kappa+\rho)}{2m_\beta(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi} \frac{\beta_k}{\sigma_k^2} \\ 0 \end{bmatrix}, \text{ and}$$

$$\mathbf{g}_k := \begin{bmatrix} -\frac{(\kappa-1)\left(m_{\alpha\beta}(1+r)^2-(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi\right)}{2m_\beta(1+r)(r+\kappa)(2r-\rho)(r-\kappa-\rho)\phi} \frac{\beta_k}{\sigma_k^2} \\ \frac{\kappa-1}{m_\beta} \frac{\beta_k}{\sigma_k^2} \end{bmatrix}.$$

Moreover, when we plug the expressions of the state variables into (OA.41), we get a stationary linear transfer process

$$Y_t = \sum_k \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right],$$

with linear filter

$$u_k(\tau) = F_k e^{-(r-\rho)\tau} + G_k e^{-\kappa\tau},$$

where

$$\begin{aligned} F_k &:= \frac{m_\beta(r+\kappa)(r-\kappa-\rho)}{2m_\beta(r+\kappa)(r-\kappa-\rho)\phi} \frac{\alpha_k}{\sigma_k^2} + \frac{m_{\alpha\beta}(\kappa-1)(1+\kappa+\rho)}{2m_\beta(r+\kappa)(r-\kappa-\rho)\phi} \frac{\beta_k}{\sigma_k^2}, \text{ and} \\ G_k &:= \frac{(\kappa-1)\left(m_{\alpha\beta}(1+r) + (\kappa+r)(\kappa-r+\rho)\phi\right)}{2m_\beta(r+\kappa)(\kappa-r+\rho)\phi} \frac{\beta_k}{\sigma_k^2}. \end{aligned}$$

The equilibrium action for the agent is stationary, and given by

$$c'(A_t) = \frac{\Delta + m_{\alpha\beta}(\kappa-1)(r+\kappa)(2r-\rho)\phi}{2m_\beta(r+\kappa)^2(2r-\rho)\phi}.$$

Thus, the contract (A, Y) just defined is an optimal stationary linear contract for the principal.

Note that, as $\rho \to 0$,

$$c'(A_t) \to \frac{\kappa-1}{4(\kappa+r)m_\beta} \left(2m_{\alpha\beta} + \sqrt{\Delta/r} \right),$$

and $\{u_k\}_k$ converges to the linear filter associated with the market belief of the conjectured optimal rating, derived in the first half of this proof.

Back to the Original Model

Now, we connect the auxiliary model and the original model, and conclude the verification. We prove by contradiction that the candidate rating obtained in the first half of this proof is indeed optimal. We continue to consider the auxiliary model. Let (A^*, Y^*) be the incentive-compatible contract defined by

$$c'(A^*) = \frac{\kappa-1}{4(\kappa+r)m_\beta} \left(2m_{\alpha\beta} + \sqrt{\Delta/r} \right),$$

and

$$Y_t^* = \frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\right)}{2\sqrt{\Delta}m_\beta(\kappa-r)} \cdot \sum_{k=1}^{K} \int_{s\le t} u_k^c(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s\right].$$

The market belief Y_t^* is defined as the conjectured optimal rating of the original setting, while A^* is the corresponding equilibrium action. Consider an information structure $\widehat{\mathcal{M}}$, generated by some rating process, that induces a stationary action $\widehat{A}$. Let $\widehat{Y} \coloneqq \mathbf{E}[\theta_t \mid \widehat{\mathcal{M}}_t]$. Note that $(\widehat{A}, \widehat{Y})$ is a well-defined incentive-compatible stationary linear contract. We show that $c'(A^*) \ge c'(\widehat{A})$. Let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined above, as a function of the principal's discount rate ρ. Let $V^{(\rho)}$ be the corresponding principal's expected payoff. Note that, for every confidential exclusive information structure $\mathcal{M}$ generated by a rating process, the equilibrium market belief of the original setting, $\mu_t = \mathbf{E}[\theta_t \mid \mathcal{M}_t]$, satisfies $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$, and thus the principal's expected payoff for contract (A^*, Y^*) is $V^* \coloneqq c'(A^*)/\rho$, while the principal's expected payoff for contract $(\widehat{A}, \widehat{Y})$ is $\widehat{V} \coloneqq c'(\widehat{A})/\rho$.

Then, for every $\rho \in (0, r)$, the inequalities $\rho V^{(\rho)} \ge \rho\widehat{V} = c'(\widehat{A})$ must hold. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*. Thus, $\mathbf{Cov}[Y^{(\rho)}, \nu_t] - \mathbf{Var}[Y^{(\rho)}] \to 0$, which in turn implies that $\rho V^{(\rho)} \to c'(A^*)$. Hence, $c'(A^*) \ge c'(\widehat{A})$.

OA.6 Proof of Corollary 4.2

In this section, we prove the following (which is more general than Corollary 4.2, because it applies to the general, multisignal version of the model, and it includes the case of public ratings):

Proposition OA.1 *Suppose the rater has access to a source of independent noise. Let $\mathcal{M}$ be either confidential (resp. public), in the sense described in Section B, and let A be the (stationary) effort level it induces. For all $A' \in [0, A]$, there exists a confidential (resp. public) information structure $\mathcal{M}'$ that induces effort level A'.*

To show that any action in the range $[0, A]$ can be attained in the equilibrium of an alternative confidential/public information structure, we modify the rating process that achieves A to depress incentives to any desired extent. To do so, we use a source of independent noise. In addition to the K signals described in the general model of Section 5, we include one additional signal, indexed by $K+1$, which is entirely uninformative about both the worker's action and the worker's ability. Let us assume S_{K+1} is a two-sided standard Brownian motion. Consider the two-sided process

$$\xi_t = \int_{s\le t} e^{-(t-s)}\,\mathrm{d}S_{K+1,s}.$$

From Lemma B.3, if Y has linear filter $\{u_k\}_k$, the equilibrium action A in both the public and confidential cases is the solution to

$$c'(A) = \frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau.$$

Consider the alternative rating process $\widehat{Y} = (1-a)Y + a\xi$, for some $a \in [0,1]$, which is a well-defined rating process for the information generated by the $K+1$ signals. Consider the information structure generated by the rating process $\widehat{Y}$, and the induced equilibrium action, $\widehat{A}$. We have

$$\begin{aligned} c'(\widehat{A}) &= \frac{\mathbf{Cov}[\widehat{Y}_t, \theta_t]}{\mathbf{Var}[\widehat{Y}_t]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau \\ &= \frac{(1-a)\,\mathbf{Cov}[Y_t, \theta_t]}{(1-a)^2\,\mathbf{Var}[Y_t] + a^2\,\mathbf{Var}[\xi]} \sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(\tau) e^{-r\tau} \, \mathrm{d}\tau \\ &= \frac{1-a}{(1-a)^2 + a^2\,\mathbf{Var}[\xi]/\mathbf{Var}[Y_t]} c'(A). \end{aligned}$$

By varying a over the interval $[0,1]$, $c'(\widehat{A})$ covers the entire interval $[0, c'(A)]$, and thus $\widehat{A}$ covers the interval $[0, A]$. Besides, as Y and ξ are independent, for $\tau \geq 0$,

$$\mathbf{Cov}[\widehat{Y}_t, \widehat{Y}_{t+\tau}] = (1-a)^2\,\mathbf{Cov}[Y_t, Y_{t+\tau}] + a^2\,\mathbf{Cov}[\xi_t, \xi_{t+\tau}].$$

By Itô's isometry, we get

$$\mathbf{Cov}[\xi_t, \xi_{t+\tau}] = \int_0^\infty e^{-s} e^{-(s+\tau)} \, \mathrm{d}s = \frac{1}{2} e^{-\tau} = \mathbf{Var}[\xi_t] e^{-\tau}.$$

Moreover, by Proposition B.9, if Y is proportional to the the belief associated with a public information structure, $\mathbf{Cov}[Y_t, Y_{t+\tau}] = \mathbf{Var}[Y_t] e^{-\tau}$. Thus,

$$\mathbf{Cov}[\widehat{Y}_t, \widehat{Y}_{t+\tau}] = ((1-a)^2\,\mathbf{Var}[Y_t] + a^2\,\mathbf{Var}[\xi_t]) e^{-\tau} = \mathbf{Var}[\widehat{Y}_t] e^{-\tau}.$$

Invoking Proposition B.9 again, we find that $\widehat{Y}$ is proportional to the market belief of a public information structure. Hence, $\widehat{A}$ also denotes the equilibrium action under that structure. In conclusion, under both the public and confidential information structure, any action in $[0, A]$ can be induced in equilibrium.

OA.7 Proof of Theorem 4.3

First, we provide the explicit specification of the functions appearing in the statement of Theorem 4.3. Define

$$c_1(t) = \frac{(r^2-1)(\kappa^2-1)}{2\sqrt{R(t)}},$$

and

$$c_2(t) = \frac{(\kappa^2-1)(\kappa+1)e^{\kappa t}\left((\kappa-1)^2(r-1)e^{-rt} + e^{\kappa t}\left(\sqrt{R(t)} - \left(\kappa^2-1\right)(1+r)\right)\right)}{2\sqrt{R(t)}\left((\kappa+1)^2 e^{2\kappa t} - (\kappa-1)^2\right)},$$

as well as

$$c_3(t) = \frac{(\kappa^2-1)(\kappa-1)\left((\kappa+1)^2(r-1)e^{-rt} + e^{-\kappa t}\left(\sqrt{R(t)} - \left(\kappa^2-1\right)(1+r)\right)\right)}{4\sqrt{R(t)}\left((\kappa^2+1)\sinh(\kappa t) + 2\kappa\cosh(\kappa t)\right)},$$

where

$$\begin{aligned}
R(t) \;&=\; \frac{(\kappa-1)^2\left((1-r)^2(r-\kappa)^2 e^{-2rt} - (1+r)^2(r+\kappa)^2\right)(\coth(\kappa t)-1)}{2r} \\
&+\; \frac{(\kappa+1)^2(r+1)^2(r-\kappa)^2(\coth(\kappa t)+1)}{2r} + 4\kappa\left(\kappa^2-1\right)\left(r^2-1\right)e^{-rt}\operatorname{csch}(\kappa t) \\
&-\; \frac{(\kappa+1)^2(1-r)^2(\kappa+r)^2 e^{2t(\kappa-r)}(\coth(\kappa t)-1)}{2r}.
\end{aligned}$$

Then the optimal rating that is referred to in Theorem 4.3 is given by

$$u_{s,t} = c_1(t)e^{-r(t-s)} + c_2(t)e^{-\kappa(t-s)} + c_3(t)e^{\kappa(t-s)}.$$

Turning, to the proof, the rater designs a linear rating Y, where

$$Y_t = \int_0^t u_{s,t}\,\mathrm{d}X_s$$

to maximize

$$\int_0^T A_s\,\mathrm{d}s,$$

where A is the worker's equilibrium effort strategy given the rating Y, and $s \mapsto u_{s,t}$ is twice continuously differentiable. The smoothness assumption on the linear filter u is used to pin down a unique candidate rating (up to scaling and shifting).

It is convenient to impose, without loss, that the rating Y is equal to the market belief, up to an additive constant. Proposition B.1 (which is stated for the case of stationary linear ratings to fit the main framework of this paper, but that remains valid for the nonstationary case) remains valid: Y is equal to a market belief, up to an additive constant, if and only if $\mathbf{Var}[Y_t] = \mathbf{Cov}[Y_t, \theta_t]$. Using Itô's Isometry, as in the proof of the baseline confidential case explained in Section OA.5, we have that $\mathbf{Var}[Y_t] = \mathbf{Cov}[Y_t, \theta_t]$ is written

$$\sigma^2 \int_0^t u_{j,t}^2 \, \mathrm{d}j + \frac{\gamma^2}{2} \int_0^t \int_0^t u_{i,t} u_{j,t} e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j = \frac{\gamma^2}{2} \int_0^t u_{j,t} e^{-|j-t|} \, \mathrm{d}j.$$

In the proof of Proposition 3.4 detailed in Section OA.3, we derived the equilibrium action of the worker for any linear rating, stationary or not. Its adaptation to the finite horizon case is immediate, and we get that

$$c'(A_s) = A_s = \int_s^T u_{s,t} e^{-r(t-s)} \, \mathrm{d}t.$$

Hence, the rater wants choose $u_{s,t}$ to maximize

$$\int_0^T A_s \, \mathrm{d}s = \int_0^T \int_s^T u_{s,t} e^{-r(t-s)} \, \mathrm{d}t \, \mathrm{d}s = \int_0^T \int_0^t u_{s,t} e^{-r(t-s)} \, \mathrm{d}s \, \mathrm{d}t,$$

subject to the constraint

$$\sigma^2 \int_0^t u_{j,t}^2 \, \mathrm{d}j + \frac{\gamma^2}{2} \int_0^t \int_0^t u_{i,t} u_{j,t} e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j = \frac{\gamma^2}{2} \int_0^t u_{j,t} e^{-|j-t|} \, \mathrm{d}j,$$

for every t.

Observe that the objective is separable in t, so that the problem for the rater reduces to maximizing, for every t,

$$\int_0^t u_{s,t} e^{rs} \, \mathrm{d}s,$$

by choosing $u_{s,t}$ such that

$$\sigma^2 \int_0^t u_{j,t}^2 \, \mathrm{d}j + \frac{\gamma^2}{2} \int_0^t \int_0^t u_{i,t} u_{j,t} e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j = \frac{\gamma^2}{2} \int_0^t u_{j,t} e^{-|j-t|} \, \mathrm{d}j. \tag{OA.43}$$

Fix some t. We consider the relaxed problem for the rater, which internalizes the variance-covariance constraint in the objective: the rater seeks to maximize

$$\int_0^t u_{s,t} e^{rs} \, \mathrm{d}s + \lambda\sigma^2 \int_0^t u_{j,t}^2 \, \mathrm{d}j + \lambda\frac{\gamma^2}{2} \int_0^t \int_0^t u_{i,t} u_{j,t} e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j - \lambda\frac{\gamma^2}{2} \int_0^t u_{j,t} e^{-|j-t|} \, \mathrm{d}j,$$

where λ is the Lagrange multiplier.

To simplify notation, we momentarily abandon the subscript t from the control variable, so that $u_{s,t}$ is now written u_s. The first-order condition, that is, the Euler-Lagrange necessary condition adapted to our setting by Proposition OA.1 of Part VI of this Online Appendix, yields

$$e^{rs} + 2\lambda\sigma^2 u_s + \lambda\gamma^2 \int_0^t u_j e^{-|j-s|}\,\mathrm{d}j - \lambda\frac{\gamma^2}{2}e^{s-t} = 0, \tag{OA.44}$$

and the Legendre second order condition is

$$\lambda < 0. \tag{OA.45}$$

We differentiate twice (with respect to s), and get

$$r^2 e^{rs} + 2\lambda\sigma^2 u_s'' - 2\lambda\gamma^2 u_s + \lambda\gamma^2 \int_0^t u_j e^{-|j-s|}\,\mathrm{d}j - \lambda\frac{\gamma^2}{2}e^{s-t} = 0. \tag{OA.46}$$

Subtracting (OA.46) from (OA.44) yields the following ODE that u_s must satisfy:

$$(1-r^2)e^{rs} + 2\lambda\sigma^2 u_s - 2\lambda\sigma^2 u_s'' + 2\lambda\gamma^2 u_s = 0.$$

Letting $\kappa = \sqrt{1+\gamma^2/\sigma^2}$, the general solution of the above ODE is

$$u_s = C^- e^{-\lambda s} + C^+ e^{\kappa s} + \frac{1-r^2}{2\lambda\sigma^2(r^2-\kappa^2)}e^{-\Delta s}.$$

It remains to identify the constants. To do so, we use the second-order condition (OA.45), the first-order condition (OA.44), and the constraint (OA.43), where $\gamma^2 = \sigma^2(\kappa^2-1)$. Fixing λ, the first-order condition (OA.44) yields a linear system of two equations in C^+ and C^-, which allows to express C^+ and C^- as a function of λ. Plugging the resulting function u_s into (OA.43), we obtain a quadratic equation in λ, with only one positive root. The details of these expressions are lengthy and omitted.

This yields a unique candidate for the optimal rating, whose general form is described in the statement of Theorem 4.3. It is tedious but straightforward to verify that the coefficients are those given above.

Part IV

Complements for Section 5

OA.8 Proof of Theorem 5.2 (and Th. B.10)

We prove Theorems 5.2 and B.10. The constants m_α, m_β, $m_{\alpha\beta}$ and κ are defined in Section B.3.1 of Appendix B.

The proof proceeds as in the proof of the baseline case of Section OA.5. The first half of the proof derives a candidate optimal rating, and the second half is self-contained and verifies that the candidate rating just derived is optimal.

OA.8.1 Candidate Optimal Rating

Recall the shorthand notation of Section OA.5 that will be used here as well:

$$U(t) \coloneqq \sum_{k=1}^{K} \beta_k u_k(t), \qquad V(t) \coloneqq \sum_{k=1}^{K} \alpha_k u_k(t),$$
$$U_0 \coloneqq \int_0^\infty U(t) e^{-t}\,\mathrm{d}t, \qquad V_0 \coloneqq \int_0^\infty V(t) e^{-rt}\,\mathrm{d}t.$$

We want to maximize $c'(A)$, with A the stationary equilibrium action of the agent, among all *public* information structures generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}[Y_t] = 1$ and that is proportional to the market belief. Such rating processes can be described by their linear filter $\mathbf{u} \coloneqq \{u_k\}_k$, and are written as

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

As in Section OA.5, we note that, by Itô's isometry, for $\tau \ge 0$,

$$\begin{aligned}\mathbf{Cov}[Y_t, Y_{t+\tau}] = {}& \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s) u_k(s+\tau)\,\mathrm{d}s \\ & + \sum_{k=1}^{K}\sum_{k'=1}^{K} \int_{i \le t}\int_{j \le t+\tau} \beta_k \beta_{k'} u_k(t-i) u_{k'}(t+\tau-j)\,\mathbf{Cov}[\theta_i, \theta_j]\,\mathrm{d}j\,\mathrm{d}i.\end{aligned}$$

Hence, as $\mathbf{Cov}[\theta_i, \theta_j] = \gamma^2 e^{-|i-j|}/2$, after a change of variables in the last term, we get

$$\mathbf{Cov}[Y_t, Y_{t+\tau}] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s) u_k(s+\tau)\,\mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

By Proposition B.9, the rating process Y is proportional to the belief of a public information structure if, and only if, $\mathbf{Cov}[Y_t, Y_{t+\tau}] = e^{-\tau}$ for every $\tau \geq 0$.

Using the expression for $\mathbf{Cov}[Y_t, Y_{t+\tau}]$ just obtained, and applying Lemma B.3, our optimization problem is thus that of maximizing

$$\frac{\gamma^2}{2} \left[\int_0^\infty U(t) e^{-t}\,\mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt}\,\mathrm{d}t \right],$$

subject to the continuum of constraints

$$\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(j) u_k(j+\tau)\,\mathrm{d}j + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j = e^{-\tau},$$

for every $\tau \geq 0$.

The continuum of constraints makes it difficult to solve this optimization problem directly by forming the Lagrangian, as we have done for the proof of confidential case. Instead, we solve a relaxed optimization problem with a single constraint: we maximize $F(\mathbf{u})$, defined as

$$F(\mathbf{u}) = \left[\int_0^\infty U(t) e^{-t}\,\mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt}\,\mathrm{d}t \right],$$

(as before, the original objective without the constant factor $\gamma^2/2$), subject to $G(\mathbf{u}) = \frac{2}{1+r}$, where

$$G(\mathbf{u}) \coloneqq g(\mathbf{u}, 0) + (1-r) \int_0^\infty e^{-r\tau} g(\mathbf{u}, \tau)\,\mathrm{d}\tau,$$

with

$$g(\mathbf{u}, \tau) \coloneqq \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(j) u_k(j+\tau)\,\mathrm{d}j + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

Assume there exists a solution $\mathbf{u}^*$ to this optimization problem, where $\mathbf{u}^*$ is twice differentiable, integrable, and square integrable. As will be shown, the solution of this relaxed constrained problem satisfies the original continuum of constraints.

As in the confidential setting, we work with an unconstrained problem that internalizes the above constraint. Thus, we relax the problem a second time, and we let

$$L(\mathbf{u}, \lambda_0) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u})$$

be the Lagrangian, from which we remove the additive terms that do not depend on $\mathbf{u}$. Assume there exists some $\lambda_0^* < 0$ such that $\mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda_0^*)$.[2] As in the confidential case, we apply Proposition OA.1 to get first-order conditions, namely: if $\lambda_0 = \lambda_0^*$ and $\mathbf{u} = \mathbf{u}^*$, then for all $k = 1, \ldots, K$ and all t, $L_k(t) = 0$, with

$$L_k(t) \coloneqq F_k(t) + \lambda_0 G_k(t),$$

$$F_k(t) \coloneqq \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t},$$

and

$$\begin{aligned} G_k(t) \coloneqq\, & 2\sigma_k^2 u_k(t) + \gamma^2 \beta_k \int_0^\infty U(j) e^{-|j-t|}\, \mathrm{d}j \\ & + (1-r)\sigma_k^2 \int_0^\infty e^{-r\tau} \left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau \\ & + (1-r)\frac{\gamma^2 \beta_k}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(j) e^{-|j+\tau-t|}\, \mathrm{d}j\, \mathrm{d}\tau \\ & + (1-r)\frac{\gamma^2 \beta_k}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(i) e^{-|t+\tau-i|}\, \mathrm{d}i\, \mathrm{d}\tau. \end{aligned}$$

Throughout the proof, any function h defined on the nonnegative real line is extended to the entire real line with the convention that these functions assign value zero to any negative input. By convention, the derivative of h at 0 is defined to be the right-derivative of h at 0, which is well-defined for h twice differentiable. Let some function $h : \mathbf{R}_+ \to \mathbf{R}$ be twice differentiable and such that h, h', h'' are all integrable. Throughout the proof, to compute derivatives of integral functions, we use the following arguments.

First, if $H(t) = \int_0^\infty h(i) e^{-|t+\tau-i|}\, \mathrm{d}i$ for some $\tau \geq 0$, then

$$H(t) = \int_0^{t+\tau} h(i) e^{-(t+\tau-i)}\, \mathrm{d}i + \int_{t+\tau}^\infty h(i) e^{t+\tau-i}\, \mathrm{d}i,$$

so that

$$H''(t) = H(t) - 2h(t+\tau).$$

Similarly, if instead $H(t) = \int_0^\infty h(j) e^{-|j+\tau-t|}\, \mathrm{d}j$ then if $t > \tau$,

$$H(t) = \int_0^{t-\tau} h(j) e^{(j+\tau-t)}\, \mathrm{d}j + \int_{t-\tau}^\infty h(j) e^{-(j+\tau-t)}\, \mathrm{d}j,$$

and for every t,

$$H''(t) = H(t) - 2h(t-\tau).$$

[2] As in the confidential setting, it is easily seen that $\lambda_0^* < 0$ is a necessary second-order condition.

Finally, if $H(t) = \int_0^\infty e^{-r\tau}\left[h(t+\tau) + h(t-\tau)\right] \mathrm{d}\tau$, then

$$H'(t) = e^{-rt}h(0) + \int_0^\infty e^{-r\tau}\left[h'(t+\tau) + h'(t-\tau)\right] \mathrm{d}\tau,$$

and

$$H''(t) = -re^{-rt}h(0) + e^{-rt}h'(0) + \int_0^\infty e^{-r\tau}\left[h''(t+\tau) + h''(t-\tau)\right] \mathrm{d}\tau.$$

We can now compute $p_k \coloneqq L_k - L_k''$ as

$$\begin{aligned}
p_k(t) = L_k(t) - L_k''(t) = \alpha_k U_0(1-r^2)e^{-rt} & \\
&+ 2\lambda_0\sigma_k^2[u_k(t) - u_k''(t)] + 2\lambda_0\gamma^2\beta_k U(t) \\
&+ \lambda_0(1-r)\sigma_k^2 \int_0^\infty e^{-r\tau}[u_k(t+\tau) + u_k(t-\tau)] \,\mathrm{d}\tau \\
&- \lambda_0(1-r)\sigma_k^2 \int_0^\infty e^{-r\tau}[u_k''(t+\tau) + u_k''(t-\tau)] \,\mathrm{d}\tau \\
&- \lambda_0(1-r)\sigma_k^2\left[-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right] \\
&+ \lambda_0(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\tau}U(t-\tau)\,\mathrm{d}\tau \\
&+ \lambda_0(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Next, we let

$$\begin{aligned}
J_k(t) &= \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau, \text{ and} \\
J(t) &= \sum_{k=1}^K \beta_k J_k(t).
\end{aligned}$$

We observe that $J_k'' = -2ru_k + r^2 J_k$. Plugging J_k in the expression for p_k:

$$\begin{aligned}
p_k(t) = \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2\beta_k U(t) & \\
+ 2r\lambda_0(1-r)\sigma_k^2 u_k(t) + \lambda_0(1-r)(1-r^2)\sigma_k^2 J_k(t) + \lambda_0(1-r)\gamma^2\beta_k J(t)&.
\end{aligned}$$

After differentiation, we get

$$\begin{aligned}
p_k''(t) = r^2\alpha U_0(1-r^2)e^{-rt} &+ 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] + 2\lambda_0\gamma^2\beta U''(t) \\
&+ 2r\lambda_0(1-r)\sigma_k^2 u_k''(t) + \lambda_0(1-r)(1-r^2)\sigma_k^2\left[-2ru_k(t) + r^2 J_k(t)\right] \\
&+ \lambda_0(1-r)\gamma^2\beta_k\left[-2rU(t) + r^2 J(t)\right].
\end{aligned}$$

Finally, we let $q_k \coloneqq p_k'' - r^2 p_k$. We have

$$\begin{aligned} q_k(t) = {} & 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] - r^2 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] \\ & + 2\lambda_0\gamma^2\beta_k U''(t) - 2r^2\lambda_0\gamma^2\beta_k U(t) \\ & + 2r\lambda_0(1-r)\sigma_k^2 u_k''(t) - 2r^3\lambda_0(1-r)\sigma_k^2 u_k(t) \\ & - 2r\lambda_0(1-r)(1-r^2)\sigma_k^2 u_k(t) \\ & - 2r\lambda_0(1-r)\gamma^2\beta_k U(t). \end{aligned}$$

We must have $q_k(t) = 0$ for all k and all t. In particular, and since $\lambda_0 \neq 0$,

$$\frac{1}{2\lambda_0}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2} q_k(t) = 0,$$

hence

$$U'' - U'''' - r^2(U - U'') + \gamma^2 m_\beta U'' - r^2\gamma^2 m_\beta U + r(1-r)U'' - r(1-r)U - r(1-r)\gamma^2 m_\beta U = 0.$$

The characteristic polynomial associated with this homogeneous linear differential equation has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$ and $\pm\sqrt{r}$. As we have assumed that the solution to the optimization problem is admissible, it follows that U must be bounded, and we discard the positive roots. Thus, U must have the form

$$U(t) = C_1 e^{-\sqrt{r}t} + C_2 e^{-\kappa t}, \tag{OA.47}$$

for some constants C_1 and C_2.

Next, pick an arbitrary pair (i, j) with $i \neq j$, and define $\zeta_{ij}(t) \coloneqq \beta_i\sigma_j^2 u_j(t) - \beta_j\sigma_i^2 u_i(t)$. That $(\beta_i q_j(t) - \beta_j q_i(t))/(2\lambda_0) = 0$ yields, after simplification, the following differential equation for ζ_{ij}:

$$\zeta_{ij}'' - \zeta_{ij}'''' - r^2(\zeta_{ij} - \zeta_{ij}'') + r(1-r)(\zeta_{ij}'' - \zeta_{ij}) = 0.$$

The characteristic polynomial associated with this homogeneous linear differential equation has roots ± 1 and $\pm\sqrt{r}$. As ζ_{ij} must be bounded, we get that ζ_{ij} has the form

$$\zeta_{ij}(t) = C_1' e^{-\sqrt{r}t} + C_2' e^{-t}, \tag{OA.48}$$

for some constants C_1' and C_2'.

Putting together (OA.47) and (OA.48), we get that

$$u_k(t) = D_{1,k} e^{-\sqrt{r}t} + D_{2,k} e^{-\kappa t} + D_{3,k} e^{-t}, \tag{OA.49}$$

for some constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$.

Determination of the Constants

We have established that the solution belongs to a family of functions that are sums of some given scaled time exponentials. We now solve for the constant factors $D_{1,k}$, $D_{2,k}$, $D_{3,k}$, $k \geq 1$.

We first note that, since the term e^{-t} vanishes in Equation (OA.47) that gives the general form of the function U, the equality

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0 \tag{OA.50}$$

obtains.

Using (OA.50), we plug (OA.49) in the equation for $L_k(t)$ and get that

$$L_k(t) = L_{1,k}e^{-rt} + L_{2,k}e^{-\kappa t} + L_{3,k}e^{-t},$$

where $L_{1,k}$, $L_{2,k}$ and $L_{3,k}$ are scalar factors that will be expressed as a function of the primitives of the model and the constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$. Note that the exponential $e^{-\sqrt{r}t}$, which appears in the general form of $u_k(t)$ given in (OA.49), vanishes after simplification, while instead an exponential e^{-rt} appears, which is not present in $u_k(t)$.

We observe that

$$\begin{aligned} L_{2,k} &= \frac{2\sigma_k^2\lambda_0(r-\kappa^2)}{(r-\kappa)(\kappa+r)}D_{2,k} + \frac{2\gamma^2\lambda_0\beta_k\left(r-\kappa^2\right)}{(\kappa-1)(\kappa+1)(\kappa-r)(\kappa+r)}\sum_{i=1}^{K}\beta_i D_{2,i} \\ &= \frac{2\lambda_0\sigma_k^2\left(r-\kappa^2\right)D_{2,k}}{(r-\kappa)(\kappa+r)} + \frac{2\lambda_0\beta_k\left(\kappa^2-r\right)}{m_\beta(r-\kappa)(\kappa+r)}\sum_{i=1}^{K}\beta_i D_{2,i}, \end{aligned}$$

using that $\gamma^2 = (\kappa^2-1)/m_\beta$. That $L_{2,k} = 0$ for all k implies

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}, \tag{OA.51}$$

for some constant a. It can be seen that if $a = 0$, then $D_{1,k} = D_{2,k} = D_{3,k} = 0$ for all k, in which case $u_k = 0$ and the variance normalization constraint is violated. Hence, in the remainder of the proof, we assume $a \neq 0$. (As it turns out, as ratings yield the same market belief up to a scalar, the precise value of a is irrelevant, as long as it is nonzero.) In particular,

$$\sum_{k=1}^{K} \alpha_k D_{2,k} = am_{\alpha\beta},$$

and

$$\sum_{k=1}^{K} \beta_k D_{2,k} = a m_\beta.$$

Using (OA.50), (OA.51), and $\gamma^2 = (\kappa^2 - 1)/m_\beta$, we get

$$\begin{aligned} L_{3,k} = {} & \frac{\left(\kappa^2 - 1\right) \lambda_0 \beta_k}{\left(\sqrt{r} - 1\right)(r+1) m_\beta} \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\beta_k}{r + \sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} \\ & + \frac{\beta_k}{r+1} \sum_{i=1}^{K} \alpha_i D_{3,i} + \frac{2\lambda_0 \sigma_k^2}{r+1} D_{3,k} + \frac{a \beta_k m_{\alpha\beta}}{\kappa + r} + \frac{a(\kappa+1)\lambda_0 \beta_k}{r+1}. \end{aligned} \tag{OA.52}$$

As $L_{3,k} = 0$ for all k, we can multiply (OA.52) by β_k/σ_k^2, sum over k, and use (OA.50) to get that $D_{3,k} = 0$. In addition, after plugging $D_{3,k} = 0$, the term $L_{3,k}$ simplifies to

$$\frac{\left(\kappa^2 - 1\right) \lambda_0 \beta_k}{\left(\sqrt{r} - 1\right)(r+1) m_\beta} \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\beta_k}{r + \sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} + \frac{a \beta_k m_{\alpha\beta}}{\kappa + r} + \frac{a(\kappa+1)\lambda_0 \beta_k}{r+1} = 0, \tag{OA.53}$$

which we will use to determine λ_0.

Finally, given $D_{2,k} = a\beta_k/\sigma_k^2$ and $D_{3,k} = 0$, and using that $\gamma^2 = (\kappa^2 - 1)/m_\beta$, the remaining constant $L_{1,k}$ simplifies to

$$\begin{aligned} L_{1,k} = {} & \left(\frac{\alpha_k}{\sqrt{r} + 1} - \frac{\left(\kappa^2 - 1\right) \lambda_0 \beta_k}{\left(\sqrt{r} - 1\right) \sqrt{r} (r+1) m_\beta} \right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda_0 \left(\sqrt{r} + 1\right) \sigma_k^2}{\sqrt{r}} D_{1,k} \\ & + \frac{a \alpha_k m_\beta}{\kappa + 1} + \frac{a \lambda_0 \beta_k (\kappa + r)}{r+1}. \end{aligned} \tag{OA.54}$$

As $L_{1,k} = 0$ must hold for every k, we multiply (OA.54) by β_k/σ_k^2, sum over k, and we get an equation that the term $\sum_i \beta_i D_{1,i}$ must satisfy:

$$\begin{aligned} & \left(\frac{m_{\alpha\beta}}{\sqrt{r} + 1} - \frac{\left(\kappa^2 - 1\right) \lambda_0 m_\beta}{\left(\sqrt{r} - 1\right) \sqrt{r} (r+1) m_\beta} \right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda_0 \left(\sqrt{r} + 1\right)}{\sqrt{r}} \sum_{i=1}^{K} \beta_i D_{1,i} \\ & \qquad + \frac{a m_{\alpha\beta} m_\beta}{\kappa + 1} + \frac{a \lambda_0 m_\beta (\kappa + r)}{r+1} = 0. \end{aligned} \tag{OA.55}$$

As $m_{\alpha\beta} \geq 0$ and $m_\beta > 0$,

$$\frac{a m_{\alpha\beta} m_\beta}{\kappa + 1} + \frac{a \lambda_0 m_\beta (\kappa + r)}{r+1} \neq 0, \tag{OA.56}$$

which implies that the factor of $\sum_i \beta_i D_{1,i}$ is nonzero. Similarly, if we multiply (OA.54) by

α_k/σ_k^2 and sum over k, we get an equation that the term $\sum_i \alpha_i D_{1,i}$ must satisfy:

$$\left(\frac{m_\alpha}{\sqrt{r}+1} - \frac{\left(\kappa^2-1\right)\lambda_0 m_{\alpha\beta}}{\left(\sqrt{r}-1\right)\sqrt{r}(r+1)m_\beta}\right)\sum_{i=1}^{K}\beta_i D_{1,i} + \frac{\lambda_0\left(\sqrt{r}+1\right)}{\sqrt{r}}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{a m_\alpha m_\beta}{\kappa+1} + \frac{a\lambda_0 m_{\alpha\beta}(\kappa+r)}{r+1} = 0. \quad \text{(OA.57)}$$

Now we can solve for $\sum_i \alpha_i D_{1,i}$ and $\sum_i \beta_i D_{1,i}$, using (OA.55) and (OA.57). Plugging in the solutions in (OA.53), we get a rational expression in λ_0, whose denominator is

$$(\kappa+1)(r+1)^2\left(\sqrt{r}-1\right)\sqrt{r}\left(\sqrt{r}+1\right)^2 m_{\alpha\beta}(\kappa+r)+(\kappa+1)\lambda_0(r+1)\left(\sqrt{r}+1\right)^3(r-\kappa)(\kappa+r)^2,$$

and whose numerator is

$$-a(r+1)^2\left(\sqrt{r}-\kappa\right)\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa+1)m_{\alpha\beta}^2(\kappa+2r-1)\right) + 4a(\kappa+1)\lambda_0(r+1)\sqrt{r}\left(\sqrt{r}+1\right)^2 m_{\alpha\beta}\left(\sqrt{r}-\kappa\right)(\kappa+r) + a(\kappa+1)^2\lambda_0^2\left(\sqrt{r}+1\right)^4\left(\sqrt{r}-\kappa\right)(\kappa+r)^2.$$

We observe that the numerator, which must equal zero, yields a quadratic equation in λ_0,

$$a\left(\sqrt{r}-\kappa\right)\left(A\lambda_0^2+B\lambda_0+C\right)=0, \quad \text{(OA.58)}$$

where:

$$\begin{aligned} A &= (\kappa+1)^2(\kappa+r)^2\left(\sqrt{r}+1\right)^4, \\ B &= 4(\kappa+1)(\kappa+r)\left(\sqrt{r}+1\right)^2\sqrt{r}(r+1)m_{\alpha\beta}, \\ C &= -(r+1)^2\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa+1)m_{\alpha\beta}^2(\kappa+2r-1)\right), \\ &= -(r+1)^2\left((\kappa+r)^2\left(m_\alpha m_\beta - m_{\alpha\beta}^2\right)+(1-r)^2 m_{\alpha\beta}^2\right). \end{aligned}$$

Next, we have that $A>0$, and also that $C<0$, which owes to the Cauchy-Schwarz inequality $m_\alpha m_\beta \geq m_{\alpha\beta}^2$ and to $\kappa>1$. Hence, there are two real roots of (OA.58), one negative, and one positive. As $B>0$ and we have established that $\lambda_0<0$, it follows that

$$\lambda_0 = \frac{-B+\sqrt{B^2-4AC}}{2A},$$

which, after simplification, reduces to

$$\lambda_0 = -\frac{(r+1)\left(\sqrt{\Delta}+2\sqrt{r}m_{\alpha\beta}\right)}{(\kappa+1)\left(\sqrt{r}+1\right)^2(\kappa+r)},$$

with $\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$.

Finally, (OA.54) and (OA.55) yield a linear equation that determines $D_{1,k}$:

$$D_{1,k} = -\frac{a\sqrt{r}(r+1)m_\beta\left(\sqrt{r}-\kappa\right)(\kappa+r)}{(\kappa+1)\left(\left(r^2-1\right)\sqrt{r}m_{\alpha\beta}+\lambda_0\left(\sqrt{r}+1\right)^2\left(r^2-\kappa^2\right)\right)}\frac{\alpha_k}{\sigma_k^2}$$
$$-\frac{a(r+1)\sqrt{r}m_{\alpha\beta}\left(\kappa+r-\sqrt{r}-1\right)+a\lambda_0\left(r-\sqrt{r}\right)\left(\sqrt{r}+1\right)^2(\kappa+r)}{\left(\left(r^2-1\right)\sqrt{r}m_{\alpha\beta}+\lambda_0\left(\sqrt{r}+1\right)^2\left(r^2-\kappa^2\right)\right)}\frac{\beta_k}{\sigma_k^2}.$$

Letting $\lambda = (\kappa-1)\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa-r)\sqrt{\Delta}$, we can make further simplifications, and express the solution in a form similar to that of the confidential case. We have

$$u_k(t) = ad_k\frac{\sqrt{r}}{\lambda}e^{-\sqrt{r}t} + a\frac{\beta_k}{\sigma_k^2}e^{-\kappa t},$$

with

$$d_k = \frac{\kappa-\sqrt{r}}{\kappa-r}c_k + \lambda\frac{\sqrt{r}-1}{\kappa-r}\frac{\beta_k}{\sigma_k^2},$$

and, as in the confidential setting,

$$c_k = (\kappa^2-r^2)m_\beta\frac{\alpha_k}{\sigma_k^2} + (1-\kappa^2)m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2}. \tag{OA.59}$$

As in the case of confidential ratings, because a rating process induces the same effort level up to scaling, all constant multipliers a yield the same effort level. We can use, for example, $a = 1$ in the preceding expressions.

If

$$a = -\frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)\left(\sqrt{r}-\kappa\right)},$$

then it can be verified that the corresponding rating process satisfies the conditions of Proposition B.9, so is a market belief for a public information structure.

OA.8.2 Verification of Optimality

We now establish that the candidate rating obtained above is indeed optimal. We continue to use the auxiliary setting introduced in Section OA.5, with the same variables and notation, except for the principal's instantaneous payoff function H. To define the principal's payoff, we introduce an additional state variable, Λ, with initial value $\Lambda_0 = 0$, and which evolves according to

$$\mathrm{d}\Lambda_t = -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t. \tag{OA.60}$$

Instead of using H as in the auxiliary setting of Section OA.5, we let

$$H_t = c'(A_t) - \phi_1 Y_t(Y_t - \nu_t) - \phi_2 Y_t \left(\frac{Y_t}{1+r} - \Lambda_t\right),$$

where

$$\phi_1 := \frac{2\sqrt{\Delta}}{(1+r)(\kappa-1)(\kappa+r)}, \text{ and } \phi_2 := \frac{\sqrt{\Delta}(r-1)}{(\kappa-1)(\kappa+r)},$$

and Δ is as defined in Section B.3.1, namely, $\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2$.

Compared to the case of the confidential setting of Section OA.5, we require two penalty terms to ensure that the principal's payoff (in the auxiliary setting) and the rater's objective (in the original setting) are comparable. As in the confidential exclusive case, the term $\phi_1 Y_t(Y_t - \nu_t)$ can be interpreted as a Lagrangian term ensuring that the optimal transfer for the principal is close to a market belief (in the sense of the original setting). The second term, $\phi_2 Y_t \left(\frac{Y_t}{1+r} - \Lambda_t\right)$, is new. It captures the public constraint: together with the first term, it ensures that the transfer for the principal is close to a market belief derived from a *public information structure*. Indeed, recall that any public market belief μ satisfies $\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Var}[\mu_t]e^{-\tau}$, by Proposition B.9. If

$$\Lambda_t = \int_0^t e^{-r(t-s)} \mu_s \, \mathrm{d}s,$$

it is immediate that Λ satisfies (OA.60) for $Y = \mu$ and

$$\mathbf{E}\left[\mu_t \Lambda_t\right] = \int_0^t e^{-r(t-s)}\, \mathbf{Cov}\left[\mu_s, \mu_t\right] \mathrm{d}s = \frac{\mathbf{Var}\left[\mu_t\right]}{1+r}\left(1 - e^{-(1+r)t}\right).$$

Thus,

$$\mathbf{E}\left[\mu_t \left(\frac{\mu_t}{1+r} - \Lambda_t\right)\right] = \frac{\mathbf{Var}\left[\mu_t\right]}{1+r} e^{-(1+r)t}.$$

As opposed to the first penalty term, this expectation does not vanish for finite values of t, because $\Lambda_0 = 0$ (more generally, as long as Λ_0 is set independently of the contract, the above expectation cannot be zero for every market belief). However, it converges exponentially to zero as t grows, and this turns out to be sufficient for our purposes. The specific values for ϕ_1 and ϕ_2 are carefully selected using the conjectured optimal rating derived from Euler-Lagrange-type necessary conditions in the first half of this proof.

The principal's problem is an optimal control problem with three state variables: the agent's estimate of his ability, ν, the state associated with the public constraint, Λ, and the agent's continuation transfer, J. The state variables evolve according to

$$\begin{aligned}
\mathrm{d}\nu_t &= -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\\
\mathrm{d}J_t &= (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\frac{\xi_\beta}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k^2} + C_k\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],\\
\mathrm{d}\Lambda_t &= -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t,
\end{aligned}$$

with $\xi_\beta \coloneqq \sum_k \beta_k C_k$ and $C_k \coloneqq \int_{\tau\geq 0} e^{-r\tau}u_k(\tau)\,\mathrm{d}\tau$. As in the confidential exclusive setting considered in Section OA.5, the principal's problem can be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$, which maximizes, for all t,

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)}\left(c'(A_s) - \phi_1 Y_s(Y_s - \nu_s) - \phi_2 Y_s\left(\frac{Y_s}{1+r} - \Lambda_s\right)\right)\mathrm{d}s \,\middle|\, \mathcal{R}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha$, where $\widehat{\xi}_\alpha \coloneqq \sum_k \alpha_k \widehat{C}_k$.
2. The evolution of the agent's belief ν, given by

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

3. The evolution of the state Λ, given by $\mathrm{d}\Lambda_t = -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t$.
4. The evolution of the agent's continuation transfer J, given by

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$

   where $\widehat{\xi}_\beta \coloneqq \sum_k \beta_k \widehat{C}_k$.
5. The transversality conditions, given by

$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau} \mid \mathcal{R}_t] = 0, \qquad \text{and} \qquad \lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau}^2 \mid \mathcal{R}_t] = 0.$$

We use dynamic programming to solve the principal's problem. The principal maximizes

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)}\left(\widehat{\xi}_{\alpha,s} - \phi_1 Y_s(Y_s - \nu_s) - \phi_2 Y_s\left(\frac{Y_s}{1+r} - \Lambda_s\right)\right)\mathrm{d}s \,\middle|\, \mathcal{R}_t\right],$$

for every t, subject to the evolution of the state variables and the transversality conditions. As before, we solve the principal's problem without imposing the restriction that transfer processes be stationary linear, and verify *ex post* that the optimal transfer in this relaxed problem is stationary linear. Assume the principal's value function V is $\mathcal{C}^2(\mathbf{R}^3)$, as a function of the states J, ν and Λ. By standard arguments, Itô's Lemma yields as HJB equation

$$\rho V = \sup_{y,c_1,\dots,c_K} \rho\widehat{\xi_\alpha} - \rho\phi_1 y(y-\nu) - \rho\phi_2 y\left(\frac{y}{1+r} - \Lambda\right) - \nu V_\nu + (rJ - y)V_J + (-r\Lambda + y)V_\Lambda + \frac{\widehat{\xi_\beta}}{m_\beta}\frac{(\kappa-1)(\kappa+r)}{1+r}V_{\nu J} + \frac{(\kappa-1)^2}{2m_\beta}V_{\nu\nu} + \frac{1}{2}\sum_{k=1}^K\left(\frac{\widehat{\xi_\beta}}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k} + \sigma_k c_k\right)^2 V_{JJ}, \tag{OA.61}$$

where, as before, to shorten notation, we have used the subscript notation for the (partial) derivatives of V, and have abused notation by using $\widehat{\xi_\alpha}$ and $\widehat{\xi_\beta}$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$V(J,\nu,\Lambda) = a_0 + a_1\nu + a_2 J + a_3\Lambda + a_4\nu J + a_5\nu\Lambda + a_6 J\Lambda + a_7\nu^2 + a_8 J^2 + a_9\Lambda^2. \tag{OA.62}$$

We plug (OA.62) into the dynamic programming equation (OA.61) and solve for the optimal control variables $y, c_1, \dots, c_K$. The equation is quadratic in $(y, c_1, \dots, c_K)$. The second-order conditions are

$$\phi_1 + \frac{\phi_2}{1+r} > 0, \text{ and} \tag{OA.63}$$

$$a_8 < 0. \tag{OA.64}$$

That condition (OA.63) is satisfied is immediate by the definition of ϕ_1 and ϕ_2. Assuming momentarily that (OA.64) holds, the first-order conditions yield as maximizers

$$y(J,\Lambda,\nu) = \frac{(a_6 - 2a_8)(r+1)}{2\rho((r+1)\phi_1 + \phi_2)}J + \frac{(r+1)(-a_6 + 2a_9 + \rho\phi_2)}{2\rho((r+1)\phi_1 + \phi_2)}\Lambda + \frac{(r+1)(-a_4 + a_5 + \rho\phi_1)}{2\rho((r+1)\phi_1 + \phi_2)}\nu + \frac{(a_3 - a_2)(r+1)}{2\rho((r+1)\phi_1 + \phi_2)}, \tag{OA.65}$$

$$c_k(J,\Lambda,\nu) = \frac{(\kappa-1)(m_{\alpha\beta}\rho(\kappa + 2r + 1) - a_4(r+1)(\kappa+r))}{2a_8 m_\beta(\kappa+r)^2} \cdot \frac{\beta_k}{\sigma_k^2} - \frac{\rho}{2a_8} \cdot \frac{\alpha_k}{\sigma_k^2}. \tag{OA.66}$$

Note that y is affine in the three state variables, and every c_k is constant. Define

$$\bar{\rho} = \sqrt{(\rho+2)(\rho+2r)}.$$

We then plug the optimal controls in (OA.61) to identify the coefficients $a_0, \dots, a_9$. Contrary to the confidential exclusive case, the system is linear-quadratic. There are two sets of coefficients that satisfy the equality (OA.61) and the second-order conditions. However, only one set of coefficients yields a state J that satisfies the transversality condition. Keeping that set of coefficients, we get:

$$a_1 = a_2 = a_3 = 0,$$

$$a_4 = \frac{\sqrt{\Delta}\rho(2r-\rho)(\rho+r+1)\left(\rho(\rho+2)+(r-1-\rho)\bar{\rho}\right)}{(\kappa-1)(\rho+2)(r-1)r(r+1)^2(\kappa+r)},$$

$$a_5 = \frac{\sqrt{\Delta}\rho^2\left((\rho+2)(r-1-\rho)(\rho+2r)+(\rho-r+1)(\rho+r+1)\bar{\rho}\right)}{(\kappa-1)(\rho+2)(r-1)r(r+1)^2(\kappa+r)},$$

$$a_6 = \frac{\sqrt{\Delta}\rho(2r-\rho)\left(2r^2-(\rho+2)r+\rho\left(\bar{\rho}-\rho-1\right)\right)}{4(\kappa-1)r^2(\kappa+r)},$$

$$a_7 = \frac{2\sqrt{\Delta}\rho(\rho-r+1)^2(\rho+r+1)^2\left(\rho-\bar{\rho}+r+1\right)}{(\kappa-1)(\rho+2)^2(r-1)^2(r+1)^4(\kappa+r)},$$

$$a_8 = -\frac{\sqrt{\Delta}\rho(2r-\rho)\left(\rho^2+\rho-\rho r+2r(r+1)\right)}{8(\kappa-1)r^2(\kappa+r)} - \frac{\sqrt{\Delta}\rho\bar{\rho}(\rho-2r)^2}{8(\kappa-1)r^2(\kappa+r)},$$

$$a_9 = \frac{\sqrt{\Delta}\rho^3\left(\rho-\bar{\rho}+r+1\right)}{8(\kappa-1)r^2(\kappa+r)}.$$

The expression for a_0 does not impact the calculations that follow. Therefore, it is omitted. Note that, if $\rho < r$, the coefficient a_8 is negative, hence (OA.64) is satisfied, and the maximizers are determined by the first-order condition. After inserting the coefficients $a_1, \dots, a_9$ into (OA.65) and (OA.66), we obtain the optimal processes $\widehat{C}_k$, which are constant (and whose expression is lengthy and omitted), as well as the optimal transfer at time t, Y_t, as a linear function of the state variables J_t, ν_t, Λ_t:

$$Y_t = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cdot \begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix}, \tag{OA.67}$$

with

$$b_1 := \frac{(2r-\rho)\left(\bar{\rho}-\rho+2r\right)}{4r}, \quad b_2 := \frac{\rho\left(\rho-\bar{\rho}+2r\right)}{4r}, \quad \text{and } b_3 := \frac{\left((\rho+1)^2-r^2\right)\left(\bar{\rho}-\rho-2\right)}{(\rho+2)(r-1)(r+1)^2}.$$

We insert the optimal control Y_t into the equations that determine the evolution of the state variables. Doing so yields a linear three-dimensional stochastic differential equation, namely

$$\mathrm{d}\begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} = M \begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \widehat{\xi}_{\beta,t} \dfrac{\gamma^2}{(1+\kappa)(1+r)} \dfrac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t} \\ 0 \\ \dfrac{\kappa - 1}{m_\beta} \dfrac{\beta_k}{\sigma_k^2} \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where

$$M := \begin{bmatrix} r - b_1 & -b_2 & -\widehat{\xi}_\beta \dfrac{\kappa + r}{1+r} - b_3 \\ b_1 & -r + b_2 & b_3 \\ 0 & 0 & -\kappa \end{bmatrix}.$$

The matrix M has three eigenvalues,

$$\delta_f := \frac{1}{4}\left(3\rho - \bar{\rho} - \sqrt{2}\sqrt{\rho(\rho + \bar{\rho} + 1) + 2r^2 - r(\rho + 2\bar{\rho} - 2)} - 2r\right),$$
$$\delta_g := \frac{1}{4}\left(3\rho - \bar{\rho} + \sqrt{2}\sqrt{\rho(\rho + \bar{\rho} + 1) + 2r^2 - r(\rho + 2\bar{\rho} - 2)} - 2r\right),$$

and $\delta_h := -\kappa$. As $\rho \to 0$, it holds that $\delta_f \to -\sqrt{r}$, and $\delta_g \to -r$. Hence, if ρ is close to zero (*i.e.*, $\rho < \rho_0$, for some $\rho_0 > 0$), the eigenvalues of the matrix M are distinct and negative. We can write

$$\begin{bmatrix} J_t \\ \Lambda_t \\ \nu_t \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k e^{\delta_f (t-s)} + \mathbf{g}_k e^{\delta_g (t-s)} + \mathbf{h}_k e^{\delta_h (t-s)}\right) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where $\mathbf{f}_k, \mathbf{g}_k$ and $\mathbf{h}_k$ are three-dimensional vectors that can be expressed in closed form (the expression for $\rho > 0$ is lengthy and omitted). From (OA.67), we get

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t - s) \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

with $u_k(\tau) := F_k e^{\delta_f \tau} + G_k e^{\delta_g \tau} + H_k e^{\delta_h \tau}$, for some constants $F_k, G_k, H_k, k = 1, \ldots, K$ that depend on the parameters of the model, and, in particular, on ρ. As $\rho \to 0$, we can simplify these constants. We obtain $G_k \to 0$, and

$$F_k \to \frac{(\kappa - 1) m_\beta \left(\sqrt{r} - \kappa\right)(\kappa + r)}{m_\beta \left(\sqrt{r} + 1\right)\sqrt{\frac{\Delta}{r}}\left(\sqrt{r} - \kappa\right)} \frac{\alpha_k}{\sigma_k^2}$$

$$+\frac{(\kappa-1)\left(\sqrt{\Delta}+(\kappa-1)m_{\alpha\beta}\left(\kappa+r-\sqrt{r}+1\right)-\sqrt{\Delta r}\right)}{m_\beta\left(\sqrt{r}+1\right)\sqrt{\frac{\Delta}{r}}\left(\sqrt{r}-\kappa\right)}\frac{\beta_k}{\sigma_k^2},$$

$$H_k \to -\frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)\left(\sqrt{r}-\kappa\right)}\frac{\beta_k}{\sigma_k^2}.$$

Also, as $\rho \to 0$,

$$\widehat{C}_k \to \frac{m_\beta(\kappa+r)^2}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)}\frac{\alpha_k}{\sigma_k^2}+\frac{(\kappa-1)\left(2\sqrt{\Delta r}-(\kappa-1)m_{\alpha\beta}(\kappa+2r+1)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)}\frac{\beta_k}{\sigma_k^2}.$$

Thus,

$$\widehat{\xi}_\alpha(=c'(A)) \to \frac{(\kappa-1)\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa-1)m_{\alpha\beta}^2(\kappa+2r+1)+2m_{\alpha\beta}\sqrt{\Delta r}\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)^2(\kappa+r)},$$

and so, after simplification,

$$c'(A_t) \to \frac{(\kappa-1)\left(1-\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)^2\right)\left(2m_{\alpha\beta}+\sqrt{\frac{\Delta}{r}}\right)}{4m_\beta(\kappa+r)}.$$

Back to the Original Model

We conclude the verification and connect the auxiliary model and the original model. The procedure is analogous to the confidential case described in Section OA.5. Let (A^*, Y^*) be the incentive-compatible contract defined by

$$c'(A_t^*) = \frac{(\kappa-1)}{4m_\beta(\kappa+r)}\left(1-\left(\frac{\sqrt{r}-1}{\sqrt{r}+1}\right)^2\right)\left(2m_{\alpha\beta}+\sqrt{\frac{\Delta}{r}}\right),$$

and

$$Y_t^* = -\frac{(\kappa-1)\left((\kappa-1)m_{\alpha\beta}(r+1)\sqrt{r}+\sqrt{\Delta}(\kappa-r)\right)}{\sqrt{\Delta}m_\beta\left(\sqrt{r}+1\right)\left(\sqrt{r}-\kappa\right)}\cdot\sum_{k=1}^{K}\int_{s\le t}u_k^p(t-s)\left[\mathrm{d}S_{k,s}-\alpha_k A_s^*\,\mathrm{d}s\right].$$

Here, Y_t^* is the market belief of the conjectured optimal rating of the original setting, and A_t^* is the conjectured optimal action. Let $\widehat{\mathcal{M}}$ be a public information structure, generated by some rating process, which induces a constant action process $\widehat{A}$. Let $\widehat{Y} := \mathbf{E}[\theta_t \mid \widehat{\mathcal{M}}_t]$. Observe that $(\widehat{A}, \widehat{Y})$ is an incentive-compatible stationary linear contract. We show that $c'(A^*) \ge c'(\widehat{A})$. For $\rho < \rho_0$, let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined above.

Let V^* be the principal's expected payoff under contract (A^*, Y^*), $\widehat{V}$ her expected payoff under $(\widehat{A}, \widehat{Y})$, and $V^{(\rho)}$ her expected payoff $(A^{(\rho)}, Y^{(\rho)})$. For every public exclusive information structure $\mathcal{M}$ generated by some rating process, the equilibrium market belief of the original setting, $\mu_t = \mathbf{E}[\theta_t \mid \mathcal{M}_t]$, satisfies $\mathbf{Cov}[\mu_t, \nu_t] = \mathbf{Var}[\mu_t]$, and $\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Var}[\mu_t]e^{-\tau}$ for all $\tau > 0$, by Proposition B.9. Thus, under (μ, A), with A the equilibrium action, the state variable Λ is expressed as $\Lambda_t = \int_0^t e^{-r(t-s)}\mu_s \, ds$, and the principal's payoff is

$$\int_0^\infty e^{-\rho t}\left(c'(A) - \phi_1 \mu_t(\mu_t - \nu_t) - \phi_2 \mu_t \left(\frac{\mu_t}{1+r} - \Lambda_t\right)\right) \mathrm{d}t = \frac{c'(A_t)}{\rho} - \phi_2 \frac{\mathbf{Var}\left[\mu_t\right]}{(1+r)(1+r+\rho)}.$$

Hence, as $\rho \to 0$, $\rho V^* \to c'(A^*)$, and $\rho\widehat{V} \to c'(\widehat{A})$. For every ρ small enough, $V^{(\rho)} \geq \widehat{V}$ must hold, because $(A^{(\rho)}, Y^{(\rho)})$ is optimal. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*. In particular, $\mathbf{Cov}[Y^{(\rho)}, \nu_t] - \mathbf{Var}[Y^{(\rho)}] \to 0$, and, for every $\tau > 0$, $\mathbf{Cov}[Y_t^{(\rho)}, Y_{t+\tau}^{(\rho)}] - \mathbf{Var}[Y_t^{(\rho)}]e^{-\tau} \to 0$. Together, these two limits imply that, as $\rho \to 0$, $\rho V^{(\rho)} - \rho V^* \to 0$. Thus, $\rho V^{(\rho)} \to c'(A^*)$, implying that $c'(A^*) \geq c'(\widehat{A})$.

OA.9 Proof of Theorem 5.3 (and Th. B.12)

As for the exclusive cases, the first half of the proof derives a candidate optimal rating obtained through first-order necessary conditions, and the second half verifies that the candidate rating just derived is optimal. Recall that the constants m_α, m_β, $m_{\alpha\beta}$ and κ are defined in Section B.3.1 of Appendix B. The constants m_α^n, m_β^n, $m_{\alpha\beta}^n$, m_α^e, m_β^e, $m_{\alpha\beta}^e$, and $\hat{\kappa}$ are defined in Section B.3.3.

OA.9.1 Candidate Optimal Rating

We continue to use the shorthand notation of the proof for the exclusive cases:

$$U(t) := \sum_{k=1}^K \beta_k u_k(t), \qquad V(t) := \sum_{k=1}^K \alpha_k u_k(t),$$
$$U_0 := \int_0^\infty U(t)e^{-t}\,\mathrm{d}t, \qquad V_0 := \int_0^\infty V(t)e^{-rt}\,\mathrm{d}t.$$

We seek to maximize the equilibrium marginal cost $c'(A)$, where A is the stationary equilibrium action of the agent, among all confidential information structures with nonexclusive signals $S_1, \ldots, S_{K_0}$ that are generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}[Y_t] = 1$ and that is proportional to the market belief. Recall that such a rating process takes the form

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \left[\mathrm{d}S_{k,s} - \alpha_k A_s \, \mathrm{d}s \right],$$

where $\mathbf{u} = \{u_k\}_k$ is the associated linear filter.

Proposition B.11, accounting for the variance normalization, yields the constraints that Y must satisfy to be proportional to the market belief: for every nonexclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t] \, \mathbf{Cov}[S_{k,t}, Y_{t+\tau}] = \mathbf{Cov}[S_{k,t}, \theta_{t+\tau}], \qquad \forall t, \forall \tau \ge 0.$$

An equivalent constraint, more convenient to work with, is as follows: for every nonexclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t] \, \mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] = \mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}], \qquad \forall t, \forall \tau \ge 0.$$

Lemma B.3 continues to hold and maximizing the equilibrium marginal cost is maximizing

$$\frac{\gamma^2}{2} \left[\int_0^\infty U(t) e^{-t} \, \mathrm{d}t \right] \left[\int_0^\infty V(t) e^{-rt} \, \mathrm{d}t \right].$$

Recall from the proof for the corresponding result in the confidential exclusive setting that the variance normalization constraint can be expressed in terms of the linear filter as

$$\sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \, \mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i) U(j) e^{-|j-i|} \, \mathrm{d}i \, \mathrm{d}j = 1.$$

As for the constraints associated with the nonexclusive signals, we have:

$$\mathbf{Cov}[\theta_t, Y_t] = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \, \mathbf{Cov}[\theta_t, \theta_s] \, \mathrm{d}s = \frac{\gamma^2}{2} \int_0^\infty U(s) e^{-s} \, \mathrm{d}s,$$

$$\mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}] = \int_0^\tau \beta_k \, \mathbf{Cov}[\theta_{t+\tau}, \theta_{t+\tau-s}] \, \mathrm{d}s = \frac{\beta_k \gamma^2}{2} \int_0^\tau e^{-s} \, \mathrm{d}s = \frac{\beta_k \gamma^2}{2} (1 - e^{-\tau}),$$

and using Itô's isometry,

$$\mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] = \frac{\gamma^2}{2} \left[\int_0^\tau u_k(s) \sigma_k^2 \, \mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^\tau \int_{j=0}^\infty U(j) e^{-|i-j|} \, \mathrm{d}i \, \mathrm{d}j \right].$$

Thus, the constraints that the linear filter $\{u_k\}_k$ must satisfy are:

$$G(\mathbf{u}) = 1,$$

and for all $\tau \geq 0$, and all $k = 1, \ldots, K_0$,

$$H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau}),$$

where we define

$$G(\mathbf{u}) = \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{i=1}^{K} \int_0^\infty \sigma_i^2 u_i(t)^2\,\mathrm{d}t,$$

$$H_k(\mathbf{u}, \tau) = \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right] \left[\sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^{\tau} \int_{j=0}^{\infty} U(j)e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j\right].$$

As in the exclusive public setting, it is difficult to solve the optimization problem directly, given the continuum of constraints. We solve a relaxed optimization problem with $1 + 2K_0$ constraints: one constraint is associated with the variance normalization, to which we append two constraints for every nonexclusive signal. We thus maximize $F(\mathbf{u})$, defined as in the exclusive setting, namely

$$F(\mathbf{u}) := \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right] \left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right],$$

subject to

$$G(\mathbf{u}) = 1,$$

$$\int_0^\infty e^{-r\tau} H_k(\mathbf{u}, \tau)\,\mathrm{d}\tau = \frac{\beta_k}{r(1+r)}, \qquad \forall k = 1, \ldots, K_0,$$

$$\int_0^\infty e^{-\hat{\kappa}\tau} H_k(\mathbf{u}, \tau)\,\mathrm{d}\tau = \frac{\beta_k}{\hat{\kappa}(1+\hat{\kappa})}, \qquad \forall k = 1, \ldots, K_0.$$

Assume there exists a solution $\mathbf{u}^* = \{u_k^*\}_k$ to the relaxed problem, where $\mathbf{u}^*$ is four times differentiable, integrable, and square integrable. It will be shown that the solution of this relaxed optimization problem satisfies the original (continuum of) constraints.

To solve the relaxed problem, we consider the Lagrangian

$$\begin{aligned} L\left(\mathbf{u}, \lambda_0, \{\lambda_{r,k}\}_{k \leq K_0}, \{\lambda_{\kappa,k}\}_{k \leq K_0}\right) := F(\mathbf{u}) &+ \lambda_0 G(\mathbf{u}) \\ &+ \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} H_i(\mathbf{u}, \tau)\,\mathrm{d}\tau \\ &+ \sum_{i=1}^{K_0} \lambda_{\kappa,i} \int_0^\infty e^{-\hat{\kappa}\tau} H_i(\mathbf{u}, \tau)\,\mathrm{d}\tau. \end{aligned}$$

Assume there exist $\lambda_0^*, \lambda_{r,1}^*, \ldots, \lambda_{r,K_0}^*, \lambda_{\kappa,1}^*, \ldots, \lambda_{\kappa,K_0}^*$ such that $\mathbf{u}^*$ maximizes

$$\mathbf{u} \mapsto L\left(\mathbf{u}, \lambda_0^*, \{\lambda_{r,k}^*\}_{k \leq K_0}, \{\lambda_{\kappa,k}^*\}_{k \leq K_0}\right).$$

Assume $\lambda_0^* < 0$.[3] We shall define these constants in such a way that there is a unique solution to the unconstrained maximization problem (up to a scalar factor), that, in addition, solves the constraints of the relaxed optimization problem.

In the sequel, we drop the star notation for simplicity. Throughout, let

$$z_k(\tau) = \sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_{i=0}^{\tau} \int_{j=0}^{\infty} U(j) e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j.$$

Applying Proposition OA.1 in Part VI of this Online Appendix, we get first-order conditions: for all k and all t, $L_k(t) = 0$, where L_k is defined as follows.

If k indexes an exclusive signal, then

$$\begin{aligned}
L_k(t) &:= U_0 \alpha_k e^{-rt} + V_0 \beta_k e^{-t} \\
&\quad + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2 \beta_k \int_0^\infty U(j) e^{-|j-t|}\,\mathrm{d}j\right) \\
&\quad + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
&\quad + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \int_0^\infty e^{-\hat{\kappa}\tau} z_i(\tau)\,\mathrm{d}\tau \\
&\quad + \frac{U_0 \beta_k \gamma^2}{2} \int_0^\infty \int_0^\tau e^{-r\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{r,i} \\
&\quad + \frac{U_0 \beta_k \gamma^2}{2} \int_0^\infty \int_0^\tau e^{-\hat{\kappa}\tau} e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau \sum_{i=1}^{K_0} \lambda_{\kappa,i}.
\end{aligned}$$

[3] The inequality is a Legendre second-order condition.

If, instead, k indexes a nonexclusive signal, then we must include two additional terms:

$$\begin{aligned}
L_k(t) \coloneqq{}& U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
&+ \lambda_0\left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j)e^{-|j-t|}\,\mathrm{d}j\right) \\
&+ \beta_k e^{-t}\sum_{i=1}^{K_0}\lambda_{r,i}\int_0^\infty e^{-r\tau}z_i(\tau)\,\mathrm{d}\tau \\
&+ \beta_k e^{-t}\sum_{i=1}^{K_0}\lambda_{\kappa,i}\int_0^\infty e^{-\hat{\kappa}\tau}z_i(\tau)\,\mathrm{d}\tau \\
&+ \frac{U_0\beta_k\gamma^2}{2}\int_0^\infty\int_0^\tau e^{-r\tau}e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau\sum_{i=1}^{K_0}\lambda_{r,i} \\
&+ \frac{U_0\beta_k\gamma^2}{2}\int_0^\infty\int_0^\tau e^{-\hat{\kappa}\tau}e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}\tau\sum_{i=1}^{K_0}\lambda_{\kappa,i} \\
&+ \lambda_{r,k}U_0\sigma_k^2\int_t^\infty e^{-r\tau}\,\mathrm{d}\tau \\
&+ \lambda_{\kappa,k}U_0\sigma_k^2\int_t^\infty e^{-\hat{\kappa}\tau}\,\mathrm{d}\tau,
\end{aligned}$$

where we note that

$$\begin{aligned}
\int_t^\infty e^{-r\tau}\,\mathrm{d}\tau &= \frac{e^{-rt}}{r}, \\
\int_t^\infty e^{-\hat{\kappa}\tau}\,\mathrm{d}\tau &= \frac{e^{-\hat{\kappa}t}}{\hat{\kappa}}, \\
\int_0^\infty\int_0^\tau e^{-r\tau}e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}t &= \left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right], \\
\int_0^\infty\int_0^\tau e^{-\hat{\kappa}\tau}e^{-|s-t|}\,\mathrm{d}s\,\mathrm{d}t &= \left[\frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}(1-\hat{\kappa}^2)} - \frac{e^{-t}}{\hat{\kappa}(1-\hat{\kappa})}\right].
\end{aligned}$$

We first obtain conditions on u_k when k indexes an exclusive signal. In the same fashion as in the public exclusive case, we obtain

$$
\begin{aligned}
L_k(t) - L_k''(t) = \alpha_k U_0 (1-r^2) e^{-rt} & \\
+ 2\lambda_0 \sigma_k^2 \left[u_k(t) - u_k''(t)\right] + 2\lambda\gamma^2\beta_k U(t) & \\
+ U_0 \beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i \gamma^2}{2} & \\
+ U_0 \beta_k \frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \frac{\beta_i \gamma^2}{2}. &
\end{aligned}
\tag{OA.68}
$$

Now, let k denote the index of a nonexclusive signal. We then obtain

$$
\begin{aligned}
L_k(t) - L_k''(t) = \alpha_k U_0 (1-r^2) e^{-rt} & \\
+ 2\lambda_0 \sigma_k^2 \left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2\beta_k U(t) & \\
+ U_0 \beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i \gamma^2}{2} & \\
+ U_0 \beta_k \frac{2e^{-\hat{\kappa}t}}{\hat{\kappa}} \sum_{i=1}^{K_0} \lambda_{\kappa,i} \frac{\beta_i \gamma^2}{2} & \\
+ (1-r^2)\lambda_{r,k} U_0 \sigma_k^2 \frac{e^{-rt}}{r} & \\
+ (1-\hat{\kappa}^2)\lambda_{\kappa,k} U_0 \sigma_k^2 \frac{e^{-\hat{\kappa}t}}{\hat{\kappa}}. &
\end{aligned}
\tag{OA.69}
$$

The equality $L_k - L_k'' = 0$ must hold for every k. Multiplying (OA.68) by $\frac{\beta_k}{2\sigma_k^2}$, and summing over the exclusive signal index k, multiplying (OA.69) by $\beta_k/(2\sigma_k^2)$ and summing over the nonexclusive signal index k, and aggregating those two summations, we get that

$$
2\lambda_0 \left[U(t) - U''(t)\right] + 2\lambda_0\gamma^2 m_\beta U(t)
$$

is equal to a sum of two exponentials with rates $-r$ and $-\hat{\kappa}$.

As by assumption, $\lambda_0 \neq 0$, the characteristic polynomial associated with this linear differential equation has the familiar roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$. As U is bounded, it implies that U, expressed as a sum of a particular solution to the above ODE and a solution to the homogeneous ODE, can be written as the sum of three exponentials:

$$
U(t) = C_1 e^{-rt} + C_2 e^{-\kappa t} + C_3 e^{-\hat{\kappa}t}. \tag{OA.70}
$$

Plugging back (OA.70) into (OA.68) and (OA.69) and equating to zero, we obtain K additional differential equations, one for every u_k. Accounting for the fact that u_k must remain bounded, the general solution of these equations yields that u_k is the sum of four exponentials, that is,

$$u_k(t) = D_{1,k}e^{-rt} + D_{2,k}e^{-\kappa t} + D_{3,k}e^{-\hat{\kappa}t} + D_{4,k}e^{-t}, \tag{OA.71}$$

for some constants $D_{1,k}, D_{2,k}, D_{3,k}, D_{4,k}$.

Determination of the Constants

We plug the general form of u_k obtained in (OA.71) in the expression for $L_k(t)$. We get that, for both exclusive and nonexclusive signal indices k, L_k can be written in the form of a sum of four exponential terms

$$L_k = L_{1,k}e^{-rt} + L_{2,k}e^{-\kappa t} + L_{3,k}e^{-\hat{\kappa}t} + L_{4,k}e^{-t},$$

where, as in the exclusive cases, the constant factors $L_{1,k}$, $L_{2,k}$, $L_{3,k}$, and $L_{4,k}$ depend on the primitives of the model and the constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$ and $D_{4,k}$. Asserting that $L_k = 0$ is equivalent to $L_{1,k} = L_{2,k} = L_{3,k} = L_{4,k} = 0$.

Let

$$A_i = \sum_{k=1}^{K} \alpha_k D_{i,k}, \qquad B_i = \sum_{k=1}^{K} \beta_k D_{i,k},$$
$$B_r = \sum_{k=1}^{K_0} \beta_k \lambda_{r,k}, \qquad B_\kappa = \sum_{k=1}^{K_0} \beta_k \lambda_{\kappa,k},$$

and let

$$\xi_{r,i} = \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau,$$
$$\xi_{\kappa,i} = \int_0^\infty e^{-\kappa\tau} z_i(\tau)\,\mathrm{d}\tau.$$

For a nonexclusive signal index k, we have the following:

$$L_{1,k} = \frac{\left(\left(r^2-1\right)\sigma_k^2\left(2\lambda_0 r D_{1,k} + U_0\lambda_{r,k}\right) + \gamma^2\left(-\beta_k\right)\left(2B_1\lambda_0 r + B_r U_0\right) + r\left(r^2-1\right)U_0\alpha_k\right)}{r\left(r^2-1\right)},$$
$$L_{2,k} = \frac{2\lambda_0\left(\left(\kappa^2-1\right)\sigma_k^2 D_{2,k} - B_2\gamma^2\beta_k\right)}{\kappa^2-1},$$
$$L_{3,k} = \frac{\left(\left(\hat{\kappa}^2-1\right)\sigma_k^2\left(2\hat{\kappa}\lambda_0\left(D_{3,k}+D_{4,k}\right) + U_0\lambda_{\kappa,k}\right) - \gamma^2\beta_k\left(2B_3\hat{\kappa}\lambda_0 + B_\kappa U_0\right)\right)}{\hat{\kappa}\left(\hat{\kappa}^2-1\right)},$$

$$L_{4,k} = \frac{1}{2}\beta_k \left(\gamma^2\lambda_0 \left(2\left(\frac{B_1}{r-1} + \frac{B_2}{\kappa-1} + \frac{B_3}{\hat{\kappa}-1}\right) + 2B_4 t + B_4\right)\right),$$

$$+ \frac{1}{2}\beta_k \left(\gamma^2 U_0 \left(\frac{B_\kappa}{(\hat{\kappa}-1)\hat{\kappa}} + \frac{B_r}{(r-1)r}\right) + 2\sum_{i=1}^{K_0} \lambda_{\kappa,i}\xi_{\kappa,i} + 2\sum_{i=1}^{K_0} \lambda_{r,i}\xi_{r,i} + 2V_0\right).$$

Instead, for an exclusive signal index k, we have the following:

$$L_{1,k} = \frac{\left(2\lambda_0 r\left(r^2-1\right)\sigma_k^2 D_{1,k} + \gamma^2\left(-\beta_k\right)\left(2B_1\lambda_0 r + B_r U_0\right) + r\left(r^2-1\right)U_0\alpha_k\right)}{r\left(r^2-1\right)},$$

$$L_{2,k} = \frac{2\lambda_0\left(\left(\kappa^2-1\right)\sigma_k^2 D_{2,k} - B_2\gamma^2\beta_k\right)}{\kappa^2-1},$$

$$L_{3,k} = \frac{\left(2\hat{\kappa}\left(\hat{\kappa}^2-1\right)\lambda_0\sigma_k^2\left(D_{3,k}+D_{4,k}\right) - \gamma^2\beta_k\left(2B_3\hat{\kappa}\lambda_0 + B_\kappa U_0\right)\right)}{\hat{\kappa}\left(\hat{\kappa}^2-1\right)},$$

$$L_{4,k} = \frac{1}{2}\beta_k \left(\gamma^2\lambda_0 \left(2\left(\frac{B_1}{r-1} + \frac{B_2}{\kappa-1} + \frac{B_3}{\hat{\kappa}-1}\right) + 2B_4 t + B_4\right)\right)$$

$$+ \frac{1}{2}\beta_k \left(\gamma^2 U_0 \left(\frac{B_\kappa}{(\hat{\kappa}-1)\hat{\kappa}} + \frac{B_r}{(r-1)r}\right) + 2\sum_{i=1}^{K_0} \lambda_{\kappa,i}\xi_{\kappa,i} + 2\sum_{i=1}^{K_0} \lambda_{r,i}\xi_{r,i} + 2V_0\right).$$

Similarly, H_k can be written in the form of a sum of four exponential terms and a constant:

$$H_k = H_{1,k}e^{-rt} + H_{2,k}e^{-\kappa t} + H_{3,k}e^{-\hat{\kappa}t} + H_{4,k}e^{-t} + H_{5,k},$$

where it can be shown that

$$H_{1,k} = \frac{U_0\left(B_1\gamma^2\beta_k - \left(r^2-1\right)\sigma_k^2 D_{1,k}\right)}{r\left(r^2-1\right)},$$

$$H_{2,k} = \frac{U_0\left(B_2\gamma^2\beta_k - \left(\kappa^2-1\right)\sigma_k^2 D_{2,k}\right)}{\kappa\left(\kappa^2-1\right)},$$

$$H_{3,k} = \frac{U_0\left(B_3\gamma^2\beta_k - \left(\hat{\kappa}^2-1\right)\sigma_k^2\left(D_{3,k}+D_{4,k}\right)\right)}{\hat{\kappa}\left(\hat{\kappa}^2-1\right)},$$

$$H_{4,k} = \frac{1}{4}\gamma^2 U_0\beta_k\left(-\frac{2B_1}{r-1} - \frac{2B_2}{\kappa-1} - \frac{2B_3}{\hat{\kappa}-1} - B_4(2t+3)\right),$$

$$H_{5,k} = \frac{1}{4}U_0\left(\frac{4\sigma_k^2\left(\kappa\hat{\kappa}D_{1,k} + r\left(\kappa\left(D_{3,k}+D_{4,k}\right) + \hat{\kappa}D_{2,k}\right)\right)}{\kappa\hat{\kappa}r}\right)$$

$$+ \frac{1}{4}U_0\left(\gamma^2\beta_k\left(2\left(\frac{B_1(r+2)}{r(r+1)} + \frac{B_2(\kappa+2)}{\kappa(\kappa+1)} + \frac{B_3(\hat{\kappa}+2)}{\hat{\kappa}(\hat{\kappa}+1)}\right) + 3B_4\right)\right).$$

Using that $L_{2,k} = 0$ for all k, we immediately get $D_{2,k} = a\beta_k/\sigma_k^2$ for some scalar a, and thus $B_2 = am_\beta$. Using $L_{4,k} = 0$ for all k, we get that $D_{4,k}$ is proportional to β_k/σ_k^2. As the term in e^{-t} vanishes in (OA.70), we get $B_4 = 0$, which in turn implies $D_{4,k} = 0$ for all k. Using that $H_{3,k} = 0$ for every nonexclusive signal index k, we infer that $D_{3,k}$ is also proportional to β_k/σ_k^2 for every nonexclusive signal index k. Summing the term $H_{3,k}$ over these indices, and equating to zero, we get

$$B_3 = \sum_{k=1}^{K_0} \beta_k D_{3,k}.$$

Next, for an exclusive signal index k, $L_{3,k} = 0$ implies that $D_{3,k}$ is proportional to β_k/σ_k^2 for the exclusive signal indices as well. Further,

$$\sum_{k=K_0+1}^{K} \beta_k D_{3,k} = 0$$

implies that $D_{3,k} = 0$ for exclusive signal indices k.

Similarly, $H_{1,k} = 0$ implies that

$$D_{1,k} = \frac{\beta_k}{\sigma_k^2} \frac{\gamma^2}{(r^2-1)} B_1.$$

As $L_{3,k} = 0$, we get

$$B_3 = -\frac{B_\kappa U_0}{2\hat{\kappa}\lambda_0}. \tag{OA.72}$$

Then, plugging this value of B_3 into the equation $L_{3,k} = 0$ for nonexclusive signal indices k, we get that, for some ν,

$$\lambda_{\kappa,k} = \nu \frac{\beta_k}{\sigma_k^2}.$$

Furthermore, $B_\kappa = \nu m_\beta^n$.

Finally, $\sum_{k=1}^{K_0} L_{1,k} = 0$ implies that

$$B_r = -\frac{r\left(r^2-1\right) m_{\alpha\beta}^n}{-\gamma^2 m_\beta^n + r^2 - 1},$$

and plugging B_r back into the equation $L_{1,k} = 0$ gives

$$\lambda_{r,i} = -\frac{\gamma^2 r \beta_k m_{\alpha\beta}^n}{\sigma_k^2 (r^2 - \hat{\kappa}^2)} - \frac{r\alpha_k}{\sigma_k^2}.$$

Then, computing $\sum_{k=1}^{K} L_{1,k}$ and equating to zero, we get

$$B_1 = -\frac{\left(r^2-1\right) U_0 \left(m_\alpha \left(r^2-\hat{\kappa}^2\right) + m_{\alpha\beta}^n \left(\kappa^2-r^2\right)\right)}{2\lambda_0 (r-\kappa)(\kappa+r)\left(r^2-\hat{\kappa}^2\right)},$$

and

$$D_{1,k} = -\frac{U_0 \left(\gamma^2 \beta_k m_\alpha + \alpha_k \left(r^2-\kappa^2\right)\right)}{2\lambda_0 \sigma_k^2 \left(r^2-\kappa^2\right)}.$$

Note that we have

$$U_0 = \frac{a m_\beta}{\kappa+1} + \frac{B_1}{r+1} + \frac{B_3}{\hat{\kappa}+1},$$
$$V_0 = \frac{a m_{\alpha\beta}}{\kappa+r} + \frac{A_1}{2r} + \frac{A_3}{\hat{\kappa}+r}.$$

At this stage, beside a, only two unknown variables remain: λ_0 and ν.

We plug the values of the variables obtained thus far to express the values of $L_{4,k}$, $H_{5,k}$, and G. This yields two quadratic equations in λ_0 and ν, obtained by setting $L_{4,k}$ to zero and set $H_{5,k}$ equal to G (which is equal to 1).[4]

Solving for the quadratic system of equations, we obtain, after simplification,

$$\lambda_0 = \frac{\sqrt{r}\left((\kappa+1) m_{\alpha\beta}^n (\kappa+r) - (\hat{\kappa}+1) m_{\alpha\beta} (\hat{\kappa}+r)\right)}{(\kappa+1)(\hat{\kappa}+1)\sqrt{r}(\kappa+r)(\hat{\kappa}+r)} - \frac{\sqrt{\Delta}(\kappa\hat{\kappa}-1)}{(\kappa+1)(\hat{\kappa}+1)\sqrt{r}},$$

and

$$\nu = \frac{\gamma^2 \hat{\kappa}}{\hat{\kappa}+1}\left(\frac{(r+1) m_{\alpha\beta}^n}{(r-\hat{\kappa})(\hat{\kappa}+r)} + \frac{\sqrt{\Delta}}{\sqrt{r}}\right),$$

where ν is chosen to be the unique negative root of the quadratic equation that results (the other root is the unique positive root and is associated with a minimum of the objective function), and where, as in Section B.3.1,

$$\Delta := \frac{(\kappa+1)(\hat{\kappa}+1)}{2(\kappa-\hat{\kappa})}\left[\frac{m_\alpha^e m_\beta^e}{\kappa^2-\hat{\kappa}^2} + \frac{(1+2r+\hat{\kappa})(m_{\alpha\beta}^n)^2}{(r+\hat{\kappa})^2(\hat{\kappa}+1)} - \frac{(1+2r+\kappa) m_{\alpha\beta}^2}{(r+\kappa)^2(\kappa+1)}\right].$$

(We prove below that $\Delta \geq 0$, so the square root is well-defined.)

Plugging back these expressions into the variables obtained so far, and (re)defining

$$\lambda = (\kappa-1)\left(\sqrt{r}(1+r) m_{\alpha\beta} + (\kappa^2-r^2)\sqrt{\Delta}\right),$$

[4] The expressions for $L_{4,k}$, $H_{5,k}$, and G are lengthy and therefore omitted. The details of the derivation and the two equations are available upon request.

we obtain after simplification, for a given factor a, the following variables.

For $k = 1, \ldots, K_0$:

$$D_{1,k} = -\frac{a\left(\kappa^2-1\right)\sqrt{r}\left(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2\right)+m^n_{\alpha\beta}\left(\kappa^2-r^2\right)\right)}{\lambda\left(r^2-\hat{\kappa}^2\right)}\frac{\beta_k}{\sigma_k^2},$$

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2},$$

$$D_{3,k} = a\left[\frac{(\kappa+1)}{(\hat{\kappa}+1)} - \frac{\left(\kappa^2-1\right)\sqrt{r}(r+1)\left(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2\right)+m^n_{\alpha\beta}\left(r^2-\kappa^2\right)\right)}{(\hat{\kappa}+1)\lambda\left(r^2-\hat{\kappa}^2\right)}\right]\frac{\beta_k}{\sigma_k^2},$$

and for $k = K_0+1, \ldots, K$:

$$D_{1,k} = -\frac{a\left(\kappa^2-1\right)m_{\alpha\beta}\sqrt{r}}{\lambda}\frac{\beta_k}{\sigma_k^2} - \frac{a\left(\kappa^2-1\right)\left(r^2-\kappa^2\right)\sqrt{r}}{\gamma^2\lambda}\frac{\alpha_k}{\sigma_k^2},$$

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2},$$

$$D_{3,k} = 0.$$

It can be verified that the rating process Y defined by such linear filter $\{u_k\}_k$ satisfies the initial set of constraints for every a, except for the normalization constraint; but as rating policies yield the same market belief when multiplied by a scalar, the exact value of a does not need to be determined.

To conclude, we show that $\Delta \geq 0$, or equivalently that $\delta_0 \geq 0$, with

$$\delta_0 := \frac{(\kappa+1)(r+\kappa)^2}{\kappa^2-\hat{\kappa}^2}m^e_\alpha m^e_\beta - (1+2r+\kappa)m^2_{\alpha\beta} + \frac{(\kappa+1)(r+\kappa)^2(1+2r+\hat{\kappa})}{(r+\hat{\kappa})^2(\hat{\kappa}+1)}(m^n_{\alpha\beta})^2.$$

We can express δ_0 as

$$\delta_0 = \frac{(\kappa+1)(r+\kappa)^2}{\kappa^2-\hat{\kappa}^2}(m^e_\alpha m^e_\beta - (m^e_{\alpha\beta})^2) + f((m^n_{\alpha\beta})^2, (m^e_{\alpha\beta})^2),$$

where the first term is nonnegative, by the Cauchy-Schwarz inequality, and the second term is the quadratic form

$$f((m^n_{\alpha\beta})^2, (m^e_{\alpha\beta})^2) = \begin{pmatrix}(m^n_{\alpha\beta})^2 & (m^e_{\alpha\beta})^2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}(m^n_{\alpha\beta})^2\\ (m^e_{\alpha\beta})^2\end{pmatrix},$$

where

$$
\begin{aligned}
a_{11} &= \frac{(\kappa-\hat{\kappa})\left(2r^3+(1+\kappa)(1+\hat{\kappa})(\kappa+\hat{\kappa})+4r^2(1+\kappa+\hat{\kappa})+2r(1+\kappa+\hat{\kappa})^2\right)}{(1+\hat{\kappa})(r+\hat{\kappa})^2},\\
a_{12} &= a_{21} = -(1+2r+\kappa),\\
a_{22} &= \frac{r(r+(2+r)\kappa)+(1+2r+\kappa)\hat{\kappa}^2}{\kappa^2-\hat{\kappa}^2}.
\end{aligned}
$$

To prove that this quadratic form is positive semidefinite, we need to check that the three principal minors are nonnegative. Clearly $a_{11} \geq 0$ and $a_{22} \geq 0$, and finally

$$
a_{11}a_{22} - a_{12}a_{21} = \frac{2r(1+r)^2(1+\kappa)(r+\kappa)^2}{(1+\hat{\kappa})(r+\hat{\kappa})^2(\kappa+\hat{\kappa})} \geq 0,
$$

as required.

OA.9.2 Verification of Optimality

We now verify that the candidate rating introduced above is optimal among all ratings. We continue to use the auxiliary setting introduced in Section OA.5, with the same variables and notation. However, we must redefine the principal's instantaneous payoff function, H.

To do so, we first introduce K_0 extra state variables, $\Lambda_1^r, \ldots, \Lambda_{K_0}^r$, with initial value

$$
\Lambda_{k,0}^r = \frac{1}{r}\int_{s\leq 0} e^{-r(t-s)}\,\mathrm{d}S_{k,s},
$$

and which evolve according to

$$
\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right]. \tag{OA.73}
$$

We also introduce K_0 additional state variables, $\Lambda_1^\kappa, \ldots, \Lambda_{K_0}^\kappa$, with initial value

$$
\Lambda_{k,0}^\kappa = \frac{1}{\hat{\kappa}}\int_{s\leq 0} e^{-\hat{\kappa}(t-s)}\,\mathrm{d}S_{k,s},
$$

which evolve as

$$
\mathrm{d}\Lambda_{k,t}^\kappa = -\hat{\kappa}\Lambda_{k,t}^\kappa\,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right]. \tag{OA.74}
$$

Instead of using H as in the case of confidential exclusive ratings, we redefine H as follows:

$$H_t := c'(A_t) - \phi_1 Y_t(Y_t - \nu_t) + \sum_{k=1}^{K_0} \phi_{2,k} \left(\frac{\beta_k(\kappa^2 - 1)}{2m_\beta r(1+r)} - Y_t \Lambda^r_{k,t} \right) - \sum_{k=1}^{K_0} \phi_{3,k} \left(\frac{\beta_k(\kappa^2 - 1)}{2m_\beta \hat{\kappa}(1+\hat{\kappa})} - Y_t \Lambda^\kappa_{k,t} \right),$$

where

$$\phi_1 := \sqrt{\Delta / r},$$
$$\phi_{2,k} := r\frac{\alpha_k}{\sigma_k^2} + \frac{r m^n_{\alpha\beta}(\kappa^2 - 1)}{m_\beta(r^2 - \hat{\kappa}^2)} \frac{\beta_k}{\sigma_k^2},$$
$$\phi_{3,k} := \frac{\hat{\kappa}(\kappa^2 - 1)\left(m^n_{\alpha\beta}\sqrt{r}(1+r) + \sqrt{\Delta}(r^2 - \hat{\kappa}^2) \right)}{m_\beta \sqrt{r}(1+\hat{\kappa})(r^2 - \hat{\kappa}^2)} \frac{\beta_k}{\sigma_k^2}.$$

The parameter Δ is defined as in Section B.3.1.

Compared to the case of confidential exclusive ratings, we now include $1 + 2K_0$ penalty terms. As before, these penalty terms ensure that the principal's payoff in this auxiliary setting and the rater's objective in the original setting are comparable. The (by now usual) term $\phi_1 Y_t(Y_t - \nu_t)$ is a penalty term that ensures that the optimal transfer of the principal remains close to a market belief of the original setting. To ensure that the optimal transfer is close to a market belief that incorporates all relevant information of the nonexclusive signals, we add an additional penalty term for every nonexclusive signal S_k:

$$\phi_{2,k} \left(\frac{\beta_k(\kappa^2 - 1)}{2m_\beta r(1+r)} - Y_t \Lambda^r_{k,t} \right) - \phi_{3,k} \left(\frac{\beta_k(\kappa^2 - 1)}{2m_\beta \hat{\kappa}(1+\hat{\kappa})} - Y_t \Lambda^\kappa_{k,t} \right).$$

To grasp the intuition behind this term, recall that, by Proposition B.11, for every nonexclusive signal S_k, any market belief that includes all relevant information about S_k satisfies

$$\mathbf{Cov}\left[\theta_t, S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t-\tau}\right], \ \forall \tau \geq 0, \forall t,$$

or equivalently,

$$\mathbf{Cov}\left[\theta_t, S_{k,t} - S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t} - S_{k,t-\tau}\right], \ \forall \tau \geq 0, \forall t. \tag{OA.75}$$

Note that

$$\mathbf{Cov}\left[\theta_t, S_{k,t} - S_{k,t-\tau}\right] = \frac{\beta_k \gamma^2}{2}\left(1 - e^{-\tau}\right) = \frac{\beta_k(\kappa^2 - 1)}{2m_\beta}\left(1 - e^{-\tau}\right).$$

Hence,

$$\int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)},$$

and

$$\int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})}.$$

Next, remark that

$$\begin{aligned}
&\int_0^\infty e^{-r\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau \\
&\qquad = \mathbf{E}\left[\mu_t\int_0^\infty e^{-r\tau}\left(S_{k,t}-\textstyle\int_{s\le t}\alpha_k A_s\,\mathrm{d}s - S_{k,t-\tau}+\int_{s\le t-\tau}\alpha_k A_s\,\mathrm{d}s\right)\mathrm{d}\tau\right] \\
&\qquad = \mathbf{E}\left[\mu_t\frac{1}{r}\int_{s\le t} e^{-r(t-s)}\left[\mathrm{d}S_{k,s}-\alpha_k A_s\,\mathrm{d}s\right]\right] \\
&\qquad = \mathbf{E}\left[\mu_t\Lambda^r_{k,t}\right],
\end{aligned}$$

where the second equality is obtained by change of variables and integration by part. Similarly,

$$\int_0^\infty e^{-\hat{\kappa}\tau}\,\mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right]\mathrm{d}\tau = \mathbf{E}\left[\mu_t\Lambda^\kappa_{k,t}\right].$$

So, if μ is a market belief that incorporates the information of nonexclusive signal S_k, we have

$$\mathbf{E}\left[\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - \mu_t\Lambda^r_{k,t}\right] = 0, \quad \text{and} \quad \mathbf{E}\left[\frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})} - \mu_t\Lambda^\kappa_{k,t}\right] = 0.$$

If μ was a market belief, but did not incorporate the information of nonexclusive signal S_k, then these expectations would, in general, be nonzero. The factors $\phi_{2,k}$ and $\phi_{3,k}$ are chosen so as to induce the principal to choose, as optimal transfer, a market belief that does incorporate such nonexclusive information.[5]

[5] In other words, the factor

$$\phi_{2,k}e^{-r\tau} - \phi_{3,k}e^{-\hat{\kappa}\tau}$$

corresponds to an infinitesimal Lagrangian multiplier associated with the constraint of nonexclusivity

$$\mathbf{Cov}\left[\theta_t, S_{k,t}-S_{k,t-\tau}\right] = \mathbf{Cov}\left[\mu_t, S_{k,t}-S_{k,t-\tau}\right].$$

In the remainder of this proof, we will also use the following notation:

$$\Phi_{2,\alpha} = \sum_{k=1}^{K_0} \alpha_k \phi_{2,k} = r m_\alpha^n + \frac{r(m_{\alpha\beta}^n)^2(\kappa^2-1)}{m_\beta(r^2-\hat{\kappa}^2)},$$
$$\Phi_{2,\beta} = \sum_{k=1}^{K_0} \beta_k \phi_{2,k} = r m_{\alpha\beta}^n + \frac{r m_{\alpha\beta}^n m_\beta^n(\kappa^2-1)}{m_\beta(r^2-\hat{\kappa}^2)},$$
$$\Phi_{3,\alpha} = \sum_{k=1}^{K_0} \alpha_k \phi_{3,k} = \frac{\hat{\kappa}(\kappa^2-1) m_{\alpha\beta}^n \left(m_{\alpha\beta}^n \sqrt{r}(1+r) + \sqrt{\Delta}(r^2-\hat{\kappa}^2)\right)}{m_\beta \sqrt{r}(1+\hat{\kappa})(r^2-\hat{\kappa}^2)},$$
$$\Phi_{3,\beta} = \sum_{k=1}^{K_0} \beta_k \phi_{3,k} = \frac{\hat{\kappa}(\kappa^2-1) m_\beta^n \left(m_{\alpha\beta}^n \sqrt{r}(1+r) + \sqrt{\Delta}(r^2-\hat{\kappa}^2)\right)}{m_\beta \sqrt{r}(1+\hat{\kappa})(r^2-\hat{\kappa}^2)}.$$

The principal's problem is then an optimal control problem with the following natural state variables: the agent's estimate of his ability, ν, the agent's continuation transfer J, and the states associated with the nonexclusive constraints. We have the following equations for the evolution of the state variables:

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],$$
$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\frac{\xi_\beta}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k^2} + C_k\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$
$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1,\ldots,K_0,$$
$$\mathrm{d}\Lambda_{k,t}^\kappa = -\hat{\kappa}\Lambda_{k,t}^\kappa\,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1,\ldots,K_0.$$

Recall that, as in the exclusive cases, $\xi_\beta := \sum_k \beta_k C_k$ and $C_k := \int_{\tau\geq 0} e^{-r\tau}u_k(\tau)\,\mathrm{d}\tau$.

As in the verification part for the confidential exclusive setting, detailed in Section OA.5, the principal's problem can be restated as follows: the principal seeks to find a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1,\ldots,K$, such that, for all t, the principal maximizes

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)} H_s\,\mathrm{d}s \,\middle|\, \mathcal{R}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha := \sum_k \alpha_k \widehat{C}_k$.

2. The evolution of the agent's belief ν,

$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$

3. The evolution of the agent's continuation transfer J,

$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$

with $\widehat{\xi}_\beta \coloneqq \sum_k \beta_k \widehat{C}_k$.

4. The evolution of the states Λ_k^r and Λ_k^κ, for $k = 1, \dots, K_0$,

$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1, \dots, K_0,$$
$$\mathrm{d}\Lambda_{k,t}^\kappa = -\hat{\kappa}\Lambda_{k,t}^\kappa\,\mathrm{d}t + \frac{1}{\hat{\kappa}}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1, \dots, K_0.$$

5. The following transversality conditions

$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau} \mid \mathcal{R}_t] = 0, \text{ and}$$
$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau}^2 \mid \mathcal{R}_t] = 0.$$

We use dynamic programming to solve the principal's problem. The principal maximizes the expected value of

$$\int_t^\infty \rho e^{-\rho(s-t)}\Bigg(\widehat{\xi}_{\alpha,s} - \phi_1 Y_t(Y_t - \nu_t) + \sum_{k=1}^{K_0}\phi_{2,k}Y_t\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - \Lambda_{k,t}^r\right) - \sum_{k=1}^{K_0}\phi_{3,k}Y_t\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta\hat{\kappa}(1+\hat{\kappa})} - \Lambda_{k,t}^{\hat{\kappa}}\right)\Bigg),$$

conditional on $\mathcal{R}_t$, for every t, subject to the evolution of the different state variables and the transversality conditions. As in the proof of the exclusive cases, we solve the principal's problem without imposing the restriction that transfer processes be stationary linear, and verify that the optimal transfer in this relaxed problem is indeed stationary linear.

Assume the principal's value function V is jointly twice continuously differentiable, as a function of all the state variables. The Hamilton-Jacobi-Bellman (HJB) equation for V

reads

$$\begin{aligned}
\rho V = \sup_{y,c_1,\ldots,c_K} & \rho\widehat{\xi}_\alpha - \rho\phi_1 y(y-\nu) - \rho y \sum_{k=1}^{K_0} \left(\phi_{2,k}\Lambda^r_{k,t} - \phi_{3,k}\Lambda^\kappa_{k,t}\right) \\
& + V^J(rJ-y) - V^\nu \nu + \sum_{k=1}^{K_0} V^r_k \left(-r\Lambda^r_k + \frac{\beta_k}{r}\nu\right) + \sum_{k=1}^{K_0} V^\kappa_k \left(-\hat{\kappa}\Lambda^\kappa_k + \frac{\beta_k}{\hat{\kappa}}\nu\right) \\
& + \frac{V^{JJ}}{2}\left(\frac{\widehat{\xi}^2_\beta}{m_\beta}\frac{(\kappa-1)(1+2r+\kappa)}{(1+r)^2} + \sum_{k=1}^{K} \sigma^2_k c^2_k\right) + \frac{V^{J\nu}\xi_\beta(\kappa-1)(\kappa+r)}{m_\beta(1+r)} \\
& + \frac{V^{\nu\nu}(\kappa-1)^2}{2m_\beta} + \sum_{k=1}^{K_0}\left(\frac{V^{Jr}_k}{r} + \frac{V^{J\kappa}_k}{\hat{\kappa}}\right)\left(\frac{\xi_\beta(\kappa-1)\beta_k}{m_\beta(1+r)} + \sigma^2_k c_k\right) \\
& + \sum_{k=1}^{K_0}\left(\frac{V^{\nu r}_k}{r} + \frac{V^{\nu\kappa}_k}{\hat{\kappa}}\right)\frac{(\kappa-1)\beta_k}{m_\beta} + \sum_{k=1}^{K_0}\left(\frac{V^{rr}_{kk}}{2r^2} + \frac{V^{r\kappa}_{kk}}{r\hat{\kappa}} + \frac{V^{\kappa\kappa}_{kk}}{2\hat{\kappa}^2}\right)\sigma^2_k.
\end{aligned} \tag{OA.76}$$

To shorten notation, we have used the following superscript/subscript notation for the (partial) derivatives of V. We use superscripts to denote the variables $(\nu, J, \Lambda^r, \Lambda^\kappa)$, and subscripts to denote the index of the variables Λ^r and Λ^κ. For example,

$$V^{JJ} := \frac{\partial^2 V}{\partial J^2}, \; V^{Jr}_k := \frac{\partial^2 V}{\partial J \partial \Lambda^r_k}, \quad \text{and} \quad V^{r\kappa}_{kj} := \frac{\partial^2 V}{\partial \Lambda^r_k \partial \Lambda^\kappa_j}.$$

We have also abused notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$\begin{aligned}
V(J,\nu,\Lambda^r,\Lambda^\kappa) = & a_0 + a^J J + a^\nu \nu + a^{J\nu} J\nu + a^{JJ} J^2 + a^{\nu\nu}\nu^2 \\
& + \sum_{k=1}^{K_0}\left(a^r_k\Lambda^r_k + a^{Jr}_k J\Lambda^r_k + a^{\nu r}_k \nu\Lambda^r_k + a^\kappa_k \Lambda^\kappa_k + a^{J\kappa}_k J\Lambda^\kappa_k + a^{\nu\kappa}_k \nu\Lambda^\kappa_k\right) \\
& + \sum_{k,j=1}^{K_0} a^{r\kappa}_{kj}\Lambda^r_k\Lambda^\kappa_j + \sum_{1\le k\le j\le K_0}\left(a^{rr}_{kj}\Lambda^r_k\Lambda^r_j + a^{\kappa\kappa}_{kj}\Lambda^\kappa_k\Lambda^\kappa_j\right).
\end{aligned} \tag{OA.77}$$

After we substitute (OA.77) into the HJB equation (OA.76), we solve for the optimal control variables $y, c_1, \ldots, c_K$. The right-hand side of the resulting equation is a sum of two quadratic functions, one in y, the other in $(c_1, \ldots, c_K)$. These quadratic functions are strictly concave when the following second-order conditions are satisfied:

$$\phi_1 > 0 \quad \text{and} \quad a^{JJ} < 0.$$

It is immediate that the first inequality is satisfied by the definition of ϕ_1. Let us assume momentarily that the second inequality holds. The first-order conditions then yield the value of the optimal control variables, which allows to identify the constant factors of the quadratic value function. We get:

$$0 = a^J = a^\nu = a^\kappa_k = a^r_k, \quad \forall k = 1, \dots, K_0,$$

and

$$\begin{aligned}
a^{JJ} &= -\rho(2r-\rho)\phi_1, \\
a^{J\nu} &= \frac{(2r-\rho)\rho\left(2r^2\hat{\kappa}(r+\hat{\kappa})\phi_1 - \hat{\kappa}(r+\hat{\kappa})\Phi_{2,\beta} + 2r^2\Phi_{3,\beta}\right)}{2r^2(1+r)\hat{\kappa}(r+\hat{\kappa})}, \\
a^{\nu\nu} &= \rho(r-1-\rho) \\
&\quad \times \left((r-1-\rho)\phi_1^2 + d_2^{\nu\nu}\phi_1\Phi_{2,\beta} + d_3^{\nu\nu}\phi_1\Phi_{3,\beta} + d_{22}^{\nu\nu}\Phi_{2,\beta}^2 + d_{23}^{\nu\nu}\Phi_{2,\beta}\Phi_{3,\beta} + d_{33}^{\nu\nu}\Phi_{3,\beta}^2\right) \\
&\quad \times \left(4(1+r)^2(2+\rho)\phi_1\right)^{-1},
\end{aligned}$$

where

$$\begin{aligned}
d_2^{\nu\nu} &= \frac{-2r(1+r)+\rho(2+\rho)}{r^2(1+r+\rho)}, \\
d_3^{\nu\nu} &= \frac{2\left(r^2-\hat{\kappa}(1+\rho)-\rho(2+\rho)+r(3+\hat{\kappa}+\rho)\right)}{\hat{\kappa}(r+\hat{\kappa})(1+\hat{\kappa}+\rho)}, \\
d_{22}^{\nu\nu} &= \frac{8r^3(r+1)-4r^2(r+3)\rho+2\left(r+1-2r^2\right)\rho^2+(r+3)\rho^3+\rho^4}{4r^4(r-1-\rho)(r+1+\rho)(2r+\rho)}, \\
d_{23}^{\nu\nu} &= \Big(-\rho(1+\rho)(2+\rho)(\hat{\kappa}+\rho)(\hat{\kappa}+1+\rho) - 2r^5 - 4r^4(\hat{\kappa}+2+\rho) \\
&\quad - 2r^3\left(\hat{\kappa}^2+\hat{\kappa}(4+\rho)-\rho^2+\rho+3\right) \\
&\quad + r^2\left(2\hat{\kappa}^2\rho+\hat{\kappa}(-4+\rho(5\rho+6))+\rho(1+\rho)(8+3\rho)\right) \\
&\quad + r\left(\hat{\kappa}^2(2+\rho(4+\rho))+\hat{\kappa}\rho(6+\rho(6+\rho))+\rho(1+\rho)(2+\rho)\right)\Big) \\
&\quad \times \left(\hat{\kappa}r^2(\hat{\kappa}+1+\rho)(\hat{\kappa}+r)(r-1-\rho)(r+1+\rho)(\hat{\kappa}+r+\rho)\right)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
d_{33}^{\nu\nu} &= \left(2\hat{\kappa}^2(1+\rho)^2+\hat{\kappa}\rho(2+\rho)(2+3\rho)+\rho^2(1+\rho)(2+\rho)+2r^4+4r^3(\hat{\kappa}+1)\right. \\
&\quad \left. +2r^2(\hat{\kappa}(\hat{\kappa}+4)+1)-2r\left(2\hat{\kappa}^2(1+\rho)+\hat{\kappa}(2+\rho(8+3\rho))+\rho(1+\rho)(2+\rho)\right)\right) \\
&\quad \times \left(\hat{\kappa}^2(\hat{\kappa}+1+\rho)(2\hat{\kappa}+\rho)(\hat{\kappa}+r)^2(r-1-\rho)\right)^{-1}.
\end{aligned}$$

Since $\phi_1 > 0$, we have $a^{JJ} < 0$ as long as $\rho \in (0, 2r)$. Thus, the second-order conditions are satisfied. For $k = 1, \ldots, K_0$,

$$
\begin{aligned}
a_k^{Jr} &= -\frac{\rho(2r-\rho)\phi_{2,k}}{2r}, \\
a_k^{J\kappa} &= \frac{(2r-\rho)\rho\phi_{3,k}}{r+\hat{\kappa}}, \\
a_k^{\nu r} &= \frac{\rho^2\left((r-1-\rho)\phi_1 + d_2^{\nu r}\Phi_{2,\beta} + d_3^{\nu r}\Phi_{3,\beta}\right)\phi_{2,k}}{4r(1+r)(1+r+\rho)\phi_1}, \\
a_k^{\nu\kappa} &= \frac{\rho(r-\hat{\kappa}-\rho)\left((r-1-\rho)\phi_1 + d_2^{\nu\kappa}\Phi_{2,\beta} + d_3^{\nu\kappa}\Phi_{3,\beta}\right)\phi_{3,k}}{2(1+r)(r+\hat{\kappa})(1+\hat{\kappa}+\rho)\phi_1},
\end{aligned}
$$

where

$$
\begin{aligned}
d_2^{\nu r} &= \frac{-4r^2+\rho+r\rho+\rho^2}{2r^2(2r+\rho)}, \\
d_2^{\nu r} &= \frac{3r^2+r(1+\hat{\kappa})-(1+\rho)(\hat{\kappa}+\rho)}{\hat{\kappa}(r+\hat{\kappa})(r+\hat{\kappa}+\rho)}, \\
d_2^{\nu\kappa} &= \frac{-2r^2-2r\hat{\kappa}+\rho(1+\hat{\kappa}+\rho)}{2r^2(r+\hat{\kappa}+\rho)}, \\
d_2^{\nu\kappa} &= \frac{r^2-\rho(1+\rho)+r(1+3\hat{\kappa}+\rho)-\hat{\kappa}(1+2\rho)}{\hat{\kappa}(r+\hat{\kappa})(2\hat{\kappa}+\rho)}.
\end{aligned}
$$

Finally, for $1 \le k, j \le K_0$,

$$
a_{kj}^{r\kappa} = \frac{(r-\hat{\kappa}-\rho)\rho^2\phi_{2,k}\phi_{3,j}}{4r(r+\hat{\kappa})(r+\hat{\kappa}+\rho)\phi_1},
$$

and for $1 \le k \le j \le K_0$,

$$
\begin{aligned}
a_{kj}^{rr} &= \begin{cases} \frac{\rho^3\phi_{2,k}\phi_{2,j}}{8r^2(2r+\rho)\phi_1} & \text{if } k < j, \\ \frac{\rho^3\phi_{2,k}^2}{16r^2(2r+\rho)\phi_1} & \text{if } k = j, \end{cases} \\
a_{kj}^{\kappa\kappa} &= \begin{cases} \frac{\rho(-r+\hat{\kappa}+\rho)^2\phi_{3,k}\phi_{3,j}}{2(r+\hat{\kappa})^2(2\hat{\kappa}+\rho)\phi_1} & \text{if } k < j, \\ \frac{\rho(-r+\hat{\kappa}+\rho)^2\phi_{3,k}^2}{4(r+\hat{\kappa})^2(2\hat{\kappa}+\rho)\phi_1} & \text{if } k = j. \end{cases}
\end{aligned}
$$

The constant term a_0 is unwieldy and irrelevant for the sequel. Therefore, its closed form expression is omitted.

In turn, the controls are expressed as follows. If $k > K_0$, *i.e.*, if S_k is an exclusive signal, then

$$c_k(J,\nu,\Lambda) = \frac{\alpha_k}{2(2r-\rho)\phi_1\sigma_k^2} - \frac{(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi_1\right)\beta_k}{2m_\beta(r+\kappa)^2(2r-\rho)\phi_1\sigma_k^2},$$

while if $k \leq K_0$,

$$c_k(J,\nu,\Lambda) = \frac{\alpha_k}{2(2r-\rho)\phi_1\sigma_k^2} - \frac{(\kappa-1)\left(m_{\alpha\beta}(1+2r+\kappa)-(r+\kappa)(2r-\rho)\phi_1\right)\beta_k}{2m_\beta(r+\kappa)^2(2r-\rho)\phi_1\sigma_k^2} - \frac{\hat\kappa(r+\hat\kappa)\phi_{2,k}-2r^2\phi_{3,k}}{4r^2\hat\kappa(r+\hat\kappa)\phi_1},$$

and finally,

$$y(J,\nu,\Lambda) = \begin{bmatrix} b^J \\ b^\nu \\ \mathbf{b}^r \\ \mathbf{b}^\kappa \end{bmatrix} \cdot \begin{bmatrix} J \\ \nu \\ \mathbf{\Lambda}^r \\ \mathbf{\Lambda}^\kappa \end{bmatrix},$$

where

$$\begin{aligned} b^J &= 2r-\rho, \\ b^\nu &= \frac{1-r+\rho}{2+2r} + \frac{(2r-\rho)\Phi_{2,\beta}}{4r^2(r+1)\phi_1} - \frac{(2r-\rho)\Phi_{3,\beta}}{2(r+1)\hat\kappa(r+\hat\kappa)}, \\ \mathbf{b}^r &= \frac{\rho}{4r\phi_1}\boldsymbol{\phi}_2, \\ \mathbf{b}^\kappa &= -\frac{(r-\hat\kappa-\rho)}{2(r+\hat\kappa)\phi_1}\boldsymbol{\phi}_3. \end{aligned}$$

In the above equations, $\boldsymbol{\phi}_2 \coloneqq (\phi_{2,1},\ldots,\phi_{2,K_0})$ and $\boldsymbol{\phi}_3 \coloneqq (\phi_{3,1},\ldots,\phi_{3,K_0})$.

Plugging these controls back into the equations of evolution of the state variables, we obtain a $(2+2K_0)$-dimensional stochastic differential equation:

$$\mathrm{d}\begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}_t^r \\ \mathbf{\Lambda}_t^\kappa \end{bmatrix} = M \begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}_t^r \\ \mathbf{\Lambda}_t^\kappa \end{bmatrix} + \sum_{k=1}^K \begin{bmatrix} \frac{\widehat{\xi}_{\beta,t}(\kappa-1)\beta_k}{m_\beta(1+r)\sigma_k^2} + \widehat{C}_{k,t} \\ \frac{(\kappa-1)\beta_k}{m_\beta\sigma_k^2} \\ \mathbf{e}_k/r \\ \mathbf{e}_k/\hat\kappa \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],$$

where

$$M = \begin{bmatrix} -(r-\rho) & \frac{\widehat{\xi}_\beta(\kappa+r)}{1+r} - b^\nu & -(\mathbf{b}^r)^\intercal & -(\mathbf{b}^\kappa)^\intercal \\ 0 & -\kappa & 0 & 0 \\ 0 & 0 & -r\mathbf{I}_{K_0} & 0 \\ 0 & 0 & 0 & -\hat{\kappa}\mathbf{I}_{K_0} \end{bmatrix}.$$

In these equations, $\mathbf{I}_{K_0}$ denotes the $K_0 \times K_0$ identity matrix, and $\mathbf{e}_k \in \mathbf{R}^{K_0}$ is the vector with one in the k-th component and zeros elsewhere. The matrix M has four eigenvalues, $-r, -\hat{\kappa}, -\kappa, -(r-\rho)$, which are distinct and negative for $\rho \in (0, r)$. Therefore, we can write

$$\begin{bmatrix} J_t \\ \nu_t \\ \mathbf{\Lambda}^r \\ \mathbf{\Lambda}^\kappa \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k^r e^{-r(t-s)} + \mathbf{f}_k^\kappa e^{-\kappa(t-s)} + \mathbf{f}_k^{\hat{\kappa}} e^{-\hat{\kappa}(t-s)} + \mathbf{f}_k^\rho e^{-(r-\rho)(t-s)} \right) [\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t] \, ,$$

for some $(2+2K_0)$-vectors $\mathbf{f}^r, \mathbf{f}^\kappa, \mathbf{f}^{\hat{\kappa}}, \mathbf{f}^\rho$ which can be expressed in closed form as a function of the parameters of the model (as in the public exclusive case, the expression for $\rho > 0$ is lengthy and thus omitted). It follows that

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \, [\mathrm{d}S_{k,t} - \alpha_k A_t \, \mathrm{d}t] \, ,$$

with

$$u_k(\tau) = F_k^r e^{-r\tau} + F_k^\kappa e^{-\kappa\tau} + F_k^{\hat{\kappa}} e^{-\hat{\kappa}\tau} + F_k^\rho e^{-(r-\rho)\tau}.$$

In the limit as $\rho \to 0$, each factor converges, and the first and last exponential become a single exponential with rate $-r$ and factor $F^r + F^\rho$. For nonexclusive signals S_k, $k \le K_0$, we have

$$F^r + F^\rho \to -\frac{a(\kappa^2-1)\sqrt{r}\left(m_{\alpha\beta}(r^2-\hat{\kappa}^2) + m_{\alpha\beta}^n(\kappa^2-r^2)\right)}{\lambda(r^2-\hat{\kappa}^2)} \frac{\beta_k}{\sigma_k^2},$$

$$F^\kappa \to a\frac{\beta_k}{\sigma_k^2},$$

$$F^{\hat{\kappa}} \to a\left[\frac{\kappa+1}{\hat{\kappa}+1} - \frac{(\kappa^2-1)\sqrt{r}(r+1)(m_{\alpha\beta}\left(r^2-\hat{\kappa}^2) + m_{\alpha\beta}^n(r^2-\kappa^2)\right)}{(\hat{\kappa}+1)\lambda(r^2-\hat{\kappa}^2)}\right] \frac{\beta_k}{\sigma_k^2},$$

while for the exclusive signal S_k, $k > K_0$,

$$F^r + F^\rho \to -\frac{a(\kappa^2-1)m_{\alpha\beta}\sqrt{r}}{\lambda}\frac{\beta_k}{\sigma_k^2} - \frac{am_\beta(r^2-\kappa^2)\sqrt{r}}{\lambda}\frac{\alpha_k}{\sigma_k^2},$$

$$F^{\kappa} \to a\frac{\beta_k}{\sigma_k^2},$$
$$F^{\hat{\kappa}} \to 0,$$

where the scaling factor is

$$a = -\frac{(\kappa-1)\left(m_{\alpha\beta}(1+r) - (r^2-\kappa^2)\phi_1\right)}{2m_\beta(r^2-\kappa^2)\phi_1}.$$

The limit of $\widehat{\xi}_\alpha$ as $\rho \to 0$ is

$$c'(A) = \frac{(\kappa-1)\left(m^n_{\alpha\beta}(\kappa+1)(r+\kappa) + m_{\alpha\beta}(\hat{\kappa}+1)(r+\hat{\kappa})\right)}{2m_\beta(r+\kappa)(1+\hat{\kappa})(r+\hat{\kappa})} + \frac{(\kappa-1)(\kappa-\hat{\kappa})}{2m_\beta(\hat{\kappa}+1)}\sqrt{\frac{\Delta}{r}}.$$

We observe that, as $\rho \to 0$, the value obtained for $c'(A)$ corresponds to the conjectured optimum of the original model, and the optimal transfer Y corresponds to the conjectured optimal market belief of the optimal rating of the original model, presented at the end of Part I.

OA.9.2.1 Back to the Original Model

We can now conclude the verification, in a way very similar to the confidential setting detailed in Section OA.5. We will therefore skip the details.

Let (A^*, Y^*) be the incentive-compatible contract defined by Y^* as the market belief of the conjectured optimal rating of the original setting, defined by the linear filter described at the end of Part I, and A^*_t the associated conjectured optimal action.

Let $\widehat{\mathcal{M}}$ be a confidential information structure with nonexclusive signals S_k, $k \le K_0$, associated with market belief $\widehat{Y}$ and stationary action $\widehat{A}$; $(\widehat{A}, \widehat{Y})$ is then a well-defined incentive-compatible stationary linear contract. We want to show that $c'(A^*) \ge c'(\widehat{A})$.

Let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined as the optimal solution above, as a function of the discount rate of the principal ρ, with $V^{(\rho)}$ the corresponding principal's expected payoff. Under both (A^*, Y^*) and $(\widehat{A}, \widehat{Y})$, the expectation of the penalty term in the principal's payoff vanishes. Thus, the principal's expected payoff for contract (A^*, Y^*) is $V^* := c'(A^*)/\rho$, while the principal's expected payoff for contract $(\widehat{A}, \widehat{Y})$ is $\widehat{V} := c'(\widehat{A})/\rho$. Then, for every $\rho \in (0, r)$, the inequalities $\rho V^{(\rho)} \ge \rho\widehat{V} = c'(\widehat{A})$ must hold. However, as $\rho \to 0$, $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*, as shown by the limits above. Thus, the expectation of the penalty term of the principal's payoff converges to zero, which in turn implies that $\rho V^{(\rho)} \to c'(A^*)$. Hence, $c'(A^*) \ge c'(\widehat{A})$.

OA.10 Missing Formulas for Theorem 5.4 (and Th. B.13)

The missing formulas for Theorem B.13 are

$$d^n := -\frac{\kappa-1}{\delta-1} - \frac{\Lambda_1 R_\beta(\delta+\kappa-r-1)}{z(\delta-1)\left(\delta+\kappa-R_\beta\right)},$$

$$c^e := \frac{(\delta-r)\left(m_{\alpha\beta}R_\beta+z\right)}{(r-\kappa)z}, \quad d^e := \frac{(\delta-r)(\kappa+r)m_\beta R_\beta}{\left(\kappa^2-1\right)z},$$

where

$$\begin{aligned}
\Lambda_1 &:= \frac{\lambda_1(\kappa+r)\left(\left(1-\delta^2\right)m_\beta+\left(\kappa^2-1\right)m_\beta^n\right)}{(\delta-1)m_\beta(r-\delta)},\\
R_\beta &:= \frac{(\kappa-1)\left((\delta-1)(r+1)m_\beta+(\kappa+1)m_\beta^n(r+1-\delta-\kappa)\right)}{(\delta-1)m_\beta(r-\delta)},\\
z &:= \frac{m_{\alpha\beta}\left(\left(r^2-1\right)m_\beta-\left(\kappa^2-1\right)m_\beta^n\right)}{(\delta-\kappa)m_\beta}\\
&\quad+\frac{\left(r^2-\kappa^2\right)\left(\left(\kappa^2-1\right)\lambda_1 m_\beta^n-(\delta-1)m_\beta\left((\delta+1)\lambda_1+m_{\alpha\beta}^n(r-\delta)\right)\right)}{(\delta-1)(\delta-\kappa)m_\beta(r-\delta)},
\end{aligned}$$

in terms of λ_1 and δ.

The parameter λ_1 is a function of δ, and we accordingly write $\lambda_1(\delta)$ when convenient. It holds that

$$\lambda_1 = \frac{(r-\delta)\left((\kappa-1)\sigma_\beta\left(r(\delta+\kappa+1)-\delta^2\right)+(\delta+\kappa)\left(\delta^2-\kappa r\right)\right)(A_1+A_2)}{(1-\kappa)\sigma_\beta D_1+\sigma_{\alpha\beta}(\kappa+r)D_2},$$

where

$$\begin{aligned}
A_1 &:= \left(\kappa^2-1\right)m_{\alpha\beta}^2\left((\delta+\kappa)^2-(\kappa+1)\sigma_\beta(2\delta+\kappa-1)\right)\left(\left(\kappa^2-1\right)\sigma_\beta+2\sigma_{\alpha\beta}\left(r^2-\kappa^2\right)\right),\\
A_2 &:= (\kappa+r)^2\left(x^2\sigma_\alpha m_\alpha m_\beta-(\kappa+1)\sigma_{\alpha\beta}^2 m_{\alpha\beta}^2((\delta-1)(\delta+r)(r-\kappa)+x(\delta+\kappa-r-1))\right),
\end{aligned}$$

with

$$x := (\kappa+1)\sigma_\beta(\delta+\kappa-r-1)+(\delta+\kappa)(r-\kappa).$$

The expressions for D_1 and D_2 are somewhat unwieldy. It holds that

$$
\begin{aligned}
D_1 \coloneqq\ & (\kappa-1)(\kappa+1)^2\sigma_\beta^2\left(\delta^4-r^4-2r^3+2r^2\left(2\delta^2+2\delta\kappa+\kappa^2-1\right)-2\delta^2 r(2\delta+2\kappa-1)\right)\\
& -(\kappa+1)\sigma_\beta(\delta+\kappa)\Big(\delta^3\left(\delta^2+3\delta\kappa+\kappa-1\right)+r^4(\delta-2\kappa+1)\\
&\quad +r^3(\delta(3-\kappa)-3\kappa+1)+r^2\left(\delta^2(3\kappa-1)-2\delta^3+\delta\left(4\kappa^2-\kappa+1\right)+4\kappa\left(\kappa^2-1\right)\right)\\
&\quad +\delta^2 r\left(3(1+\kappa)(1-\delta)-8\kappa^2\right)\Big)\\
& +(\delta+\kappa)^2\Big(\delta^3(2\delta\kappa+\delta+\kappa)+r^4(\delta-\kappa)+(\delta+1)r^3(\delta-\kappa)\\
&\quad +r^2\left(\delta^2(\kappa+1)-\delta^3-\delta\kappa+2\kappa^2(\kappa+1)\right)-\delta^2 r\left(\delta^2-\delta\kappa+\delta+\kappa(4\kappa+3)\right)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 \coloneqq\ & \left(\kappa^2-1\right)\sigma_\beta^2\Big((\delta-1)\delta^3(\kappa-1)-r^3\left(2(\delta+\kappa)^2+(1-\delta)(\kappa-1)\right)\\
&\quad +r^2\left(4\delta^3+\delta^2(7\kappa+1)+\delta\left(4\kappa^2+\kappa-1\right)+2\kappa\left(\kappa^2-1\right)\right)\\
&\quad +\delta^2 r\left(\delta-2\delta^2-5\delta\kappa-4\kappa^2+\kappa+1\right)\Big)\\
& +\sigma_\beta(\delta+\kappa)\Big(\delta^3\left(\delta\left(1-3\kappa^2\right)-\delta^2(\kappa+1)+(\kappa-1)\kappa\right)\\
&\quad +r^3\left(\delta^2(\kappa-1)+\delta\left(3\kappa^2-1\right)+\kappa\left(4\kappa^2+\kappa-3\right)\right)\\
&\quad +r^2\left(\delta^3(1-3\kappa)+\delta^2\left(3-9\kappa^2\right)-\delta\kappa\left(4\kappa^2+\kappa-1\right)-4\kappa^2\left(\kappa^2-1\right)\right)\\
&\quad +\delta^2 r\left(\delta^2(3\kappa+1)+5\delta\kappa^2+\delta+\kappa\left(8\kappa^2-\kappa-5\right)\right)\Big)-2(\delta+\kappa)^2(r-\kappa)\left(\delta^2-\kappa r\right)^2.
\end{aligned}
$$

Finally, regarding δ, consider the polynomial

$$\tilde{P}(z) \coloneqq b_0+b_1 z+b_2 z^2+b_3 z^3+b_4 z^4+b_5 z^5+z^6,$$

with

$$
\begin{aligned}
b_0 &\coloneqq \zeta\left(\zeta+\psi g_{\alpha\beta}\right),\\
b_1 &\coloneqq \zeta\left(2\eta_\beta+g_{\alpha\beta}\right),\\
b_2 &\coloneqq \frac{1}{2}\left(-2\eta_\beta\left(2\zeta-\eta_\beta\right)-g_{\alpha\beta}\left((4\psi-1)\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta}\right),\\
b_3 &\coloneqq -2\left(\eta_\beta^2+\zeta\right)-g_{\alpha\beta}\left(\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta},\\
b_4 &\coloneqq \frac{1}{2}\left(2\left(\eta_\beta-2\right)\eta_\beta+g_{\alpha\beta}\left(\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta}\right),\\
b_5 &\coloneqq 2\eta_\beta+g_{\alpha\beta},
\end{aligned}
$$

where $\sigma_\beta \coloneqq 1 - m^n_\beta/m_\beta$, $\sigma_\alpha \coloneqq 1 - m^n_\alpha/m_\alpha$, $\sigma_{\alpha\beta} \coloneqq 1 - m^n_{\alpha\beta}/m_{\alpha\beta}$ and

$$\begin{aligned}
\eta_\beta &\coloneqq \frac{\kappa(1-\sigma_\beta)+\sigma_\beta}{r}, \quad \zeta \coloneqq \frac{\kappa^2(1-\sigma_\beta)+\sigma_\beta}{r^2},\\
g_{\alpha\beta} &\coloneqq \frac{2(\kappa-1)(r+1)^2\chi(\chi+1)m^2_{\alpha\beta}}{r\left(\sigma_\alpha m_\alpha m_\beta(\kappa+r)^2+(\kappa-1)m^2_{\alpha\beta}\left(2(r+1)\chi-(\kappa-1)\sigma_\beta\right)\right)},\\
\psi &\coloneqq \frac{(\kappa-1)\sigma_\beta+\chi(\kappa(\chi+2)+\chi)}{2r\chi(\chi+1)}, \quad \chi \coloneqq \frac{(\kappa-1)\sigma_\beta-\sigma_{\alpha\beta}(\kappa+r)}{r+1}.
\end{aligned}$$

We show the following lemma.

Lemma OA.1 *The polynomial $\tilde{P}$ is irreducible and admits no solutions in terms of radicals. It has exactly two positive distinct roots $\tilde{\delta}_-, \tilde{\delta}_+$. Let $\delta_- = r\tilde{\delta}_-, \delta_+ = r\tilde{\delta}_+$. It holds that either $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$ or $(\delta_+^2 - r)\lambda_1(\delta_+) < 0$, but not both. The parameter δ is equal to δ_- if $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$, and to δ_+ otherwise.*

Proof. Throughout, we use the notation

$$\rho_\beta = \frac{m^n_\beta}{m_\beta}, \quad p_{\alpha\beta} = m_{\alpha\beta} - m^n_{\alpha\beta} = m^e_{\alpha\beta}, \quad p_\alpha = m_\alpha - m^n_\alpha = m^e_\alpha.$$

We rule out the trivial case $\rho_\beta = 1$.

We recall that the exponent δ is given by one of the roots (if any) of the polynomial P, defined as

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6,$$

with

$$\begin{aligned}
a_0 &= r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa+r)^2\left(m_\beta p_\alpha\left(\left(\kappa^2-1\right)\rho_\beta+1\right)+\left(\kappa^2-1\right)p^2_{\alpha\beta}\right)\right.\\
&\qquad \left.-2(\kappa-1)\kappa m_{\alpha\beta}p_{\alpha\beta}(\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)^2\right),\\
a_1 &= 2r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa+r)^2\left(m_\beta p_\alpha\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)p^2_{\alpha\beta}\right)\right.\\
&\qquad \left.+(1-\kappa)m_{\alpha\beta}p_{\alpha\beta}(2\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\right),\\
a_2 &= r\left(2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r)\left((1-\kappa)\rho_\beta\left(r^2-2\kappa(\kappa+1)+r\right)+\kappa\left(r^2+r+2\right)+\kappa^2(r+2)\right.\right.\\
&\qquad \left.-r(r+1)\right)+(\kappa+r)^2\left(m_\beta p_\alpha\left((\kappa-1)\rho_\beta+1\right)\left((\kappa-1)\rho_\beta(r-2\kappa-2)+r-2\right)\right.\\
&\qquad \left.+(\kappa-1)p^2_{\alpha\beta}\left((\kappa-1)\rho_\beta(r-2\kappa-2)-\kappa(r+2)+r-2\right)\right)\\
&\qquad \left.+(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\left(2\left(\kappa^2-1\right)\rho_\beta+r^2+\kappa(r+2)+r+2\right)\right),\\
a_3 &= r\left(2(\kappa+r)^2\left(m_\beta p_\alpha\left((1-\kappa)\rho_\beta\left((\kappa-1)\rho_\beta+\kappa+3\right)-2\right)-p^2_{\alpha\beta}\left((\kappa-1)^2\rho_\beta+\kappa^2-1\right)\right)\right.\\
&\qquad +4(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r)\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+r+1)+1\right)\\
&\qquad +2(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+2)+r^2+2(\kappa+1)r+2\right)
\end{aligned}$$

$$
\begin{aligned}
a_4 &= -(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2\left(\left(\kappa^2-1\right)\rho_\beta+2\kappa+1\right)\\
&\quad+(\kappa+r)^2\left((\kappa-1)p_{\alpha\beta}^2\left((\kappa-1)\rho_\beta+2\right)-m_\beta p_\alpha\left((\kappa-1)\rho_\beta+1\right)\left(\rho_\beta(1-\kappa)+2r-1\right)\right)\\
&\quad+2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r)\left((\kappa-1)\rho_\beta(r-\kappa)+r-2\kappa\right)\\
a_5 &= 2(\kappa+r)^2\left(m_\beta p_\alpha\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)p_{\alpha\beta}^2\right)\\
&\quad-2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(2\kappa+r+1)(\kappa+r)-2(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2(\kappa+r+1),\\
a_6 &= m_\beta p_\alpha(\kappa+r)^2-2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2.
\end{aligned}
$$

We first show that such a root exists; in fact, we show that exactly two positive roots of P exist. We break the analysis in a series of claims.

1. The polynomial P is (for general parameters) irreducible and cannot be solved by radicals.
2. The coefficients a_0 and a_6 are strictly positive.
3. The polynomial P has at most two (real) positive roots.
4. The polynomial P has at least two positive roots.

We then show that these roots yield values of different signs for the candidate λ_0 providing a criterion to select the correct root. (An alternative selection criterion, of course, is to compare the value of the objective at each of these two roots.)

Step 1: P cannot be solved. For this it suffices to use a simple (not nongeneric) example. We consider the case of three signals (the first one being the only nonexclusive one) with $\alpha_1=\alpha_2=\alpha_3=1$, $\beta_k=\sigma_k=k$, $k=1,2,3$, $\gamma=1$, $r=3$, giving (up to a multiplicative constant),

$$
P(x)=1517x^6+10356x^5+15552x^4-31388x^3-97977x^2-59688x+7154.
$$

Irreducibility can be determined using the Mathematica© programming language, using the (exact) command `IrreduciblePolynomialQ`. The GAP package RadiRoot formally determines whether a given polynomial has a solvable Galois group; applying it to this polynomial establishes that it does not, and hence, its roots cannot be expressed by radicals.

Step 2: The coefficients a_0, a_6 are positive. First, we note that

$$
p_{\alpha\beta}^2<\left(1-\rho_\beta\right)m_\beta p_\alpha, \tag{OA.78}
$$

as

$$
p_{\alpha\beta}^2-\left(1-\rho_\beta\right)m_\beta p_\alpha=\left(m_{\alpha\beta}-n_{\alpha\beta}\right)^2-\left(m_\alpha-n_\alpha\right)\left(m_\beta-n_\beta\right)\leq 0
$$

by the Cauchy-Schwarz inequality. Hence,

$$\begin{aligned} a_6 &= m_\beta p_\alpha(\kappa+r)^2 - 2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r) - (\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2 \\ &\geq \frac{p_{\alpha\beta}^2(\kappa+r)^2}{1-\rho_\beta} - (\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2 - 2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r) \\ &= \frac{\left((\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta} + p_{\alpha\beta}(\kappa+r)\right)^2}{1-\rho_\beta} > 0. \end{aligned}$$

As for a_0, we first note that we can ignore the factor $r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)$ (positive). Using also (OA.78) to minorize $m_\beta p_\alpha$ (its coefficient being positive), we have

$$\begin{aligned} &(\kappa+r)^2\left(m_\beta p_\alpha\left(\left(\kappa^2-1\right)\rho_\beta+1\right)+\left(\kappa^2-1\right)p_{\alpha\beta}^2\right) \\ &\quad - 2(\kappa-1)\kappa m_{\alpha\beta}p_{\alpha\beta}(\kappa+r+1)(\kappa+r) - (\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2(\kappa+r+1)^2 \\ &\geq -2(\kappa-1)\kappa m_{\alpha\beta}p_{\alpha\beta}(\kappa+r+1)(\kappa+r) - (\kappa-1)^2\left(\rho_\beta-1\right)m_{\alpha\beta}^2(\kappa+r+1)^2 \\ &\quad + (\kappa+r)^2\left(\frac{p_{\alpha\beta}^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)}{1-\rho_\beta}+\left(\kappa^2-1\right)p_{\alpha\beta}^2\right) \\ &= \frac{\left((\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}(\kappa+r+1)+\kappa p_{\alpha\beta}(\kappa+r)\right)^2}{1-\rho_\beta}, \end{aligned}$$

which is positive.

Step 3: The polynomial P has at most two positive roots. We establish this result by applying Descartes' rule of sign, which states that an upper bound to the number of positive roots of a polynomial is given by the number of changes of signs in the sequence of its coefficients. We show that the sequence changes signs twice at most (in fact, exactly, though we won't need this).

We define

$$\begin{aligned} \eta_\beta &= \frac{(\kappa-1)\rho_\beta+1}{r}, \\ \zeta &= \frac{\left(\kappa^2-1\right)\rho_\beta+1}{r^2}, \\ g_{\alpha\beta} &= \frac{2(\kappa-1)\left((\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}+p_{\alpha\beta}(\kappa+r)\right)\left((p_{\alpha\beta}-m_{\alpha\beta})(\kappa+r)+m_{\alpha\beta}(\kappa-1)\rho_\beta\right)}{r\left((\kappa-1)^2\left(1-\rho_\beta\right)m_{\alpha\beta}^2+m_\beta p_\alpha(\kappa+r)^2-2(\kappa-1)m_{\alpha\beta}p_{\alpha\beta}(\kappa+r)\right)}, \\ \psi &= \frac{\kappa+\frac{(\kappa-1)(r+1)m_{\alpha\beta}(\kappa+r)\left(\left(\rho_\beta-1\right)m_{\alpha\beta}+p_{\alpha\beta}\right)}{\left((\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}+p_{\alpha\beta}(\kappa+r)\right)\left(m_{\alpha\beta}\left(\kappa+r-(\kappa-1)\rho_\beta\right)-p_{\alpha\beta}(\kappa+r)\right)}+1}{2r}. \end{aligned}$$

This corresponds to the consecutive change of variables

$$
\begin{aligned}
p_{\alpha\beta} &\to \pm\frac{\sqrt{w_{\alpha\beta}}\sqrt{z_\alpha}}{\sqrt{\kappa-1}(\kappa+r)},\\
m_{\alpha\beta} &\to \frac{\sqrt{w_{\alpha\beta}}\sqrt{z_\alpha}}{\sqrt{\kappa-1}},\\
w_{\alpha\beta} &\to \frac{r g_{\alpha\beta}}{2\left(y_{\alpha\beta}\left(\kappa-r\eta_\beta\right)-1\right)\left(y_{\alpha\beta}\left(\kappa-r\eta_\beta+r+1\right)-1\right)},\\
p_\alpha &\to \frac{z_\alpha\left(w_{\alpha\beta}y_{\alpha\beta}\left(y_{\alpha\beta}\left(r\eta_\beta-\kappa\right)+2\right)+1\right)}{m_\beta(\kappa+r)^2},\\
y_{\alpha\beta} &\to \frac{\kappa^2+\kappa-r^2\psi\pm D+\left(2r\psi-\kappa\right)\left(r\eta_\beta-1\right)-r\eta_\beta-2\kappa r\psi+\kappa r+r\psi}{\left(\kappa-r\eta_\beta\right)\left((\kappa+r)(\kappa-2r\psi+r+2)-\left(r\eta_\beta-1\right)\left(\kappa-2r\psi+1\right)\right)},\\
\kappa &\to \frac{\zeta r^2-r\eta_\beta}{r\eta_\beta-1},\\
\rho_\beta &\to \frac{r\eta_\beta-1}{\kappa-1},
\end{aligned}
$$

where

$$D=(r+1)\sqrt{\left(r\eta_\beta-1\right)\left(\kappa-2r\psi+1\right)+(r\psi-1)^2}.$$

The sign in front of $p_{\alpha\beta}$ is chosen positive (negative) if $m_{\alpha\beta}$ is positive (negative), and similarly, the sign in front of D is positive if

$$\left((\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}+p_{\alpha\beta}(\kappa+r)\right)\left(m_{\alpha\beta}\left(\kappa+r-(\kappa-1)\rho_\beta\right)-p_{\alpha\beta}(\kappa+r)\right)$$

is positive. Finally, we do the change $x\to rz$.

We note for future reference that $\zeta,\eta_\beta\geq 0$, and also

$$\zeta-\eta_\beta^2=\frac{(\kappa-1)^2\left(1-\rho_\beta\right)\rho_\beta}{r^2}>0, \tag{OA.79}$$

which implies in particular that

$$\zeta+\psi^2-2\psi\eta_\beta\geq\left(\psi-\eta_\beta\right)^2>0.$$

(We occasionally take the square root of the left term in what follows.)

Up to a positive multiplicative constant (namely, a_6), P is equal to $\tilde{P}$, where

$$\tilde{P}(z)=b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+b_5z^5+z^6,$$

with

$$
\begin{aligned}
b_0 &= \zeta\left(\zeta + \psi g_{\alpha\beta}\right), \\
b_1 &= \zeta\left(2\eta_\beta + g_{\alpha\beta}\right), \\
b_2 &= \frac{1}{2}\left(-2\eta_\beta\left(2\zeta - \eta_\beta\right) - g_{\alpha\beta}\left((4\psi-1)\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta}\right), \\
b_3 &= -2\left(\eta_\beta^2 + \zeta\right) - g_{\alpha\beta}\left(\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta}, \\
b_4 &= \frac{1}{2}\left(2\left(\eta_\beta - 2\right)\eta_\beta + g_{\alpha\beta}\left(\eta_\beta + \psi\right) - |g_{\alpha\beta}|\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta}\right), \\
b_5 &= 2\eta_\beta + g_{\alpha\beta}.
\end{aligned}
$$

We already know that $b_0 \geq 0$ (since $b_0 = a_0/a_6$), and it is plain that $\operatorname{sgn} b_1 = \operatorname{sgn} b_5$. We now show the following in turn.

The coefficient b_3 is negative. (This is not entirely obvious because both ψ and $g_{\alpha\beta}$ can take either sign.) Without loss, consider the case in which $g_{\alpha\beta} < 0$ (the other case is entirely symmetrical). If $\psi > 0$, then $g_{\alpha\beta} \geq -\zeta/\psi$ (because $b_0 > 0$) and so

$$
\begin{aligned}
b_3 &= g_{\alpha\beta}\left(\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta} - \eta_\beta - \psi\right) - 2\left(\eta_\beta^2 + \zeta\right) \\
&\leq -\frac{\zeta\left(\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta} - \eta_\beta + \psi\right) + 2\psi\eta_\beta^2}{\psi},
\end{aligned}
\tag{OA.80}
$$

an expression that is readily maximized over the domain $\{\psi, \zeta, \eta_\beta : \zeta \geq \eta_\beta^2\}$, with maximum 0 at $(\psi, \zeta, \eta_\beta) = (1, 0, 0)$. If instead $\psi < 0$, the maximum of (OA.80) over $g_{\alpha\beta}$ and $\{\psi, \zeta, \eta_\beta : \zeta \geq \eta_\beta^2\}$ obtains at $g_{\alpha\beta} = 0$, and $(\psi, \zeta, \eta_\beta) = (-1, 0, 0)$, and is equal to zero as well.

We note that this implies that there can be at most one switch of sign in the sequence $\{b_0, b_1, b_2, b_3\}$ if $b_2 \leq 0$, and similarly on $\{b_3, b_4, b_5, b_6 = 1\}$ if $b_4 \leq 0$. The conclusion will follow more generally if $b_2 \geq 0$, then $b_1 > 0$ (and similarly, if $b_4 \geq 0$, then $b_5 > 0$).

If $b_2 \geq 0$, then $b_1 > 0$. Suppose not, that is, suppose $b_1 \leq 0$, or $g_{\alpha\beta} \leq -2\eta_\beta (< 0)$. Clearly then

$$
b_2 = \frac{1}{2}\left(2\eta_\beta\left(\eta_\beta - 2\zeta\right) + g_{\alpha\beta}\left(\sqrt{\zeta + \psi^2 - 2\psi\eta_\beta} + (1 - 4\psi)\eta_\beta - \psi\right)\right)
$$

is largest when ζ is smallest. Either the relevant constraint is (OA.79), and substituting for ζ into b_2 to majorize it, we are left with an expression that is easily seen to admit as maximum over

$$
\left\{(g_{\alpha\beta}, \eta_\beta, \psi) : 2\eta_\beta + g_{\alpha\beta} < 0, \eta_\beta \geq 0, \eta_\beta^2 + \psi g_{\alpha\beta}\right\}
$$

the triple $(g_{\alpha\beta}, \eta_\beta, \psi) = (-1, 1/2, 1/4)$, equal to 0. Or the constraint $\zeta + \psi g_{\alpha\beta}$ binds, and eliminating ζ, the maximum over $\{2\eta_\beta + g_{\alpha\beta} < 0, \eta_\beta \geq 0\}$ is seen to be 0, achieved at $(g_{\alpha\beta}, \eta_\beta, \psi) = (-1, 0, 0)$. Hence, $b_4 < 0$.

If $b_4 \geq 0$, then $b_5 > 0$. This case is handled exactly as the previous one: if $b_5 \leq 0$, then $g_{\alpha\beta} \leq -2\eta_\beta (< 0)$, and b_4 being decreasing in ζ, we majorize b_4 by replacing ζ by its minimum value, corresponding to one of the two constraints; we then consider the two possible binding constraints for ζ, $\zeta \geq \eta_\beta^2 > 0, \zeta + \psi g_{\alpha\beta} > 0$, and show that the maximum is 0.

Step 4: The polynomial P admits at least two positive roots. Because $a_0 > 0$ and $a_6 > 0$, it suffices to show that $P(x) < 0$ for some $x > 0$, or equivalently, $\tilde{P}(z) < 0$ for some $z > 0$. We consider the polynomial $\tilde{Q}$ given by

$$\tilde{Q}(z) = z^3 + \eta_\beta (z-1) z - \zeta.$$

Its coefficients are given by $(-\zeta, -\eta_\beta, \eta_\beta, 1)$. Hence, recalling that $\eta_\beta \geq 0, \zeta \geq 0$, it follows that $\tilde{Q}$ has at most one positive (real) root. Yet clearly, $\tilde{Q}(0) < 0$ while $\lim_{z\to\infty} \tilde{Q}(z) = +\infty$; in fact, $\tilde{Q}(\sqrt{\eta}) = \eta_\beta^2 - \zeta$, which is negative, given (OA.78). Hence, Q admits exactly one root z^* in the interval $(\sqrt{\eta}, \infty)$.

We will show that $\tilde{P}(z^*) < 0$. At z^*, it holds that $\zeta = (z^*)^3 + \eta_\beta (z^* - 1) z^*$, by definition of $\tilde{Q}$. We eliminate ζ from $\tilde{P}$ using this identity, and obtain that

$$\begin{aligned}\tilde{P}(z^*) = \frac{1}{2} z^* (1 + z^*) \Bigg(& (2(z^*)^3 + \eta_\beta((z^*-1)z^* - 2\psi) + \psi(z^*-1)z^*) g_{\alpha\beta} \\ & - |g_{\alpha\beta}| z^* (z^*+1) \sqrt{\psi^2 + \eta_\beta \left((z^*-1) z^* - 2\psi\right) + (z^*)^3} \Bigg).\end{aligned}$$

We argue that the negative last term dominates the (possibly positive) other terms appearing in the brackets. To do so, compare their magnitude by taking the difference of the squares, and compute

$$\begin{aligned}&(z^*) (z^*+1)^2 \left(\psi^2 + \eta_\beta \left((z^*-1) z^* - 2\psi\right) + (z^*)^3\right) \\ &\quad - \left((2(z^*)^3 + \eta_\beta((z^*-1)z^* - 2\psi) + \psi(z^*-1)z^*)\right)^2 \\ &= \left(2\psi - (z^*)^2 + z^*\right)^2 (\eta_\beta + z^*) \left((z^*)^2 - \eta_\beta\right),\end{aligned}$$

which is positive, because $z^* > \sqrt{\eta_\beta}$. Hence, $\tilde{P}(z^*) < 0$.

How to select the root. We have not only shown that P, or $\tilde{P}$, admits two positive roots $\underline{z} < \overline{z}$, but have found in Step 4 a value z^* such that $\underline{z} < z^* < \overline{z}$. By definition, the value of z^* is the root of a polynomial, defined in Step 4 as $\tilde{Q}$.

The rational function that gives λ_0 as a function of x is equal to

$$Q(x) = \frac{Q_1(x)Q_2(x)}{Q_3(x)},$$

where Q_k are polynomial in x, with Q_1 being cubic and Q_2 quadratic (with a negative discriminant, and hence no real roots). Applying the same change of variable to Q_1 as in Step 3, we obtain a new polynomial, which is precisely (up to a positive multiplicative constant) $\tilde{Q}$. Hence, Q changes signs at z^*/r, resulting in precisely one of the two roots being associated with a negative value for the corresponding λ_0. We omit the details. ■

OA.11 Proof of Theorem 5.4 (and Th. B.13)

We proceed as for the other three cases. The first half of the proof derives a candidate optimal rating, while the second half verifies that the candidate rating just derived is optimal. We recall that the constants m_α, m_β, $m_{\alpha\beta}$ and κ are defined in Section B.3.1, while the constants m^n_α, m^n_β, $m^n_{\alpha\beta}$, m^e_α, m^e_β, $m^e_{\alpha\beta}$, and $\hat{\kappa}$ are defined in Section B.3.3.

OA.11.1 Candidate Optimal Rating

We continue to use the shorthand notation of the proof for the exclusive cases:

$$U(t) := \sum_{k=1}^{K} \beta_k u_k(t), \qquad V(t) := \sum_{k=1}^{K} \alpha_k u_k(t),$$
$$U_0 := \int_0^\infty U(t)e^{-t}\,\mathrm{d}t, \qquad V_0 := \int_0^\infty V(t)e^{-rt}\,\mathrm{d}t.$$

We maximize $c'(A)$, with A the stationary equilibrium action of the agent, among all public information structures with nonexclusive signals $S_1, \ldots, S_{K_0}$, that are generated by some rating process Y that satisfies the variance normalization $\mathbf{Var}\,[Y_t] = 1$ and that is proportional to the market belief. We express such a rating process Y via its linear filter $\mathbf{u} = \{u_k\}_k$,

$$Y_t = \sum_{k=1}^{K} \int_0^\infty u_k(t-s)\,[\mathrm{d}S_{k,s} - \alpha_k A^*\,\mathrm{d}s]\,.$$

Proposition B.11 and Proposition B.9 yield the constraints that Y must satisfy to be proportional to the market belief. Namely:

1. The constraints of nonexclusivity: for every nonexclusive signal S_k,

$$\mathbf{Cov}[\theta_t, Y_t]\,\mathbf{Cov}[S_{k,t}, Y_{t+\tau}] = \mathbf{Cov}[S_{k,t}, \theta_{t+\tau}], \qquad \forall t, \forall \tau \geq 0,$$

or equivalently,

$$\mathbf{Cov}[\theta_t, Y_t]\,\mathbf{Cov}[(S_{k,t+\tau} - S_t), Y_{t+\tau}] = \mathbf{Cov}[(S_{k,t+\tau} - S_{k,t}), \theta_{t+\tau}], \qquad \forall t, \forall \tau \geq 0.$$

2. The constraints of publicness, together with the variance normalization:

$$\mathbf{Cov}\,[Y_t, Y_{t+\tau}] = e^{-\tau}, \qquad \forall r, \forall \tau \geq 0.$$

Recall from the proofs of the optimal ratings for the nonexclusive confidental and exclusive public cases, in Sections OA.9 and OA.8 respectively, that the constraints can be expressed in terms of the linear filter of the rating process as follows:

$$G(\mathbf{u}, \tau) = e^{-\tau} \qquad \text{and} \qquad H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau}) \qquad \forall \tau \geq 0, \forall k = 1, \ldots, K_0,$$

with

$$G(\mathbf{u}, \tau) = \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j)e^{-|j+\tau-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)u_k(s+\tau)\,\mathrm{d}s,$$
$$H_k(\mathbf{u}, \tau) = \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2} \int_{i=0}^{\tau} \int_{j=0}^{\infty} U(j)e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j\right].$$

We solve a relaxed optimization problem with $2 + K_0$ constraints: one constraint is associated with the variance normalization, one constraint associated with the constraints of public ratings, and one constraint associated with each nonexclusive signal. Specifically, we maximize $F(\mathbf{u})$, defined as

$$F(\mathbf{u}) = \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right],$$

and equal to a constant times $c'(A)$, subject to

$$G(\mathbf{u}, 0) = 1,$$
$$\int_0^\infty e^{-r\tau} G(\mathbf{u}, \tau)\,\mathrm{d}\tau = \frac{1}{1+r},$$
$$\int_0^\infty e^{-r\tau} H_k(\mathbf{u}, \tau)\,\mathrm{d}\tau = \frac{\beta_k}{r(1+r)}, \qquad \forall k = 1, \ldots, K_0.$$

Assume there exists a solution $\mathbf{u}^* = \{u_k^*\}_k$ to the above problem that is four times continuously differentiable, integrable, and square integrable, and which in addition satisfies the continuum of constraints of the original problem.

Let

$$L\left(\mathbf{u},\lambda_0,\lambda_1,\{\lambda_{r,k}\}_{k\le K_0}\right) = F(\mathbf{u}) + \lambda_0 G(\mathbf{u},0) + \lambda_1 \int_0^\infty e^{-r\tau} G(\mathbf{u},\tau)\,\mathrm{d}\tau + \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} H_i(\mathbf{u},\tau)\,\mathrm{d}\tau.$$

Assume there exist $\lambda_0^*, \lambda_1^*, \lambda_{r,1}^*, \ldots, \lambda_{r,K_0}^*$ such that $\mathbf{u}^*$ maximizes

$$\mathbf{u} \mapsto L\left(\mathbf{u}, \lambda_0^*, \lambda_1^*, \{\lambda_{r,k}^*\}_{k\le K_0}\right).$$

Assume $\lambda_1^*/\lambda_0^* > -r$. We will choose these constants in such a way that there is a unique solution to the unconstrained maximization problem (up to a scalar factor), which in addition solves the constraints of the original problem.

In the sequel, we drop the star notation for simplicity. Throughout, let

$$z_k(\tau) = \sigma_k^2 \int_0^\tau u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2} \int_{i=0}^\tau \int_{j=0}^\infty U(j) e^{-|i-j|}\,\mathrm{d}i\,\mathrm{d}j.$$

As in the other three cases, we apply Proposition OA.1 of this Online Appendix, and get the following first-order conditions: for all k and all t, it holds that $L_k(t) = 0$, where L_k is defined as follows.

If k indexes an exclusive signal, then

$$\begin{aligned}
L_k(t) \coloneqq{} & U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
& + \lambda_0 \left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k \int_0^\infty U(j) e^{-|j-t|}\,\mathrm{d}j\right) \\
& + \lambda_1\sigma_k^2 \int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(j) e^{-|j+\tau-t|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \lambda_1\beta_k\frac{\gamma^2}{2} \int_0^\infty e^{-r\tau} \int_0^\infty U(j) e^{-|t+\tau-j|}\,\mathrm{d}j\,\mathrm{d}\tau \\
& + \beta_k e^{-t} \sum_{i=1}^{K_0} \lambda_{r,i} \int_0^\infty e^{-r\tau} z_i(\tau)\,\mathrm{d}\tau \\
& + U_0\beta_k \left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right] \sum_{i=1}^{K_0} \lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}$$

If k indexes a nonexclusive signal, then

$$\begin{aligned}
L_k(t) \coloneqq{}& U_0\alpha_k e^{-rt} + V_0\beta_k e^{-t} \\
&+ \lambda_0\left(2\sigma_k^2 u_k(t) + \gamma^2\beta_k\int_0^\infty U(j)e^{-|j-t|}\,\mathrm{d}j\right) \\
&+ \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k(t+\tau)+u_k(t-\tau)\right]\mathrm{d}\tau \\
&+ \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|j+\tau-t|}\,\mathrm{d}j\,\mathrm{d}\tau \\
&+ \lambda_1\beta_k\frac{\gamma^2}{2}\int_0^\infty e^{-r\tau}\int_0^\infty U(j)e^{-|t+\tau-j|}\,\mathrm{d}j\,\mathrm{d}\tau \\
&+ \beta_k e^{-t}\sum_{i=1}^{K_0}\lambda_{r,i}\int_0^\infty e^{-r\tau}z_i(\tau)\,\mathrm{d}\tau \\
&+ U_0\beta_k\left[\frac{2e^{-rt}}{r(1-r^2)} - \frac{e^{-t}}{r(1-r)}\right]\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2} \\
&+ \lambda_{r,k}U_0\sigma_k^2\frac{e^{-rt}}{r}.
\end{aligned}$$

We first obtain conditions on u_k when k indexes an exclusive signal. In a similar fashion as in the case of public exclusive information structures, we obtain that

$$\begin{aligned}
L_k(t) - L_k''(t) ={}& \alpha_k U_0(1-r^2)e^{-rt} \\
&+ 2\lambda_0\sigma_k^2[u_k(t)-u_k''(t)] + 2\lambda_0\gamma^2\beta_k U(t) \\
&+ \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k(t+\tau)+u_k(t-\tau)\right]\mathrm{d}\tau \\
&- \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k''(t+\tau)+u_k''(t-\tau)\right]\mathrm{d}\tau \\
&- \lambda_1\sigma_k^2\left[-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right] \\
&+ \lambda_1\gamma^2\beta_k\int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau \\
&+ U_0\beta_k\frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2}.
\end{aligned}$$

Let

$$p_k(t) = L_k(t) - L_k''(t),$$

and let

$$J_k(t) = \int_0^\infty e^{-r\tau} \left[u_k(t+\tau) + u_k(t-\tau)\right] \mathrm{d}\tau,$$
$$J(t) = \sum_{i=1}^{K} \beta_i J_i(t).$$

Observe that

$$J_k''(t) = -2ru_k(t) + r^2 J_k(t),$$

and, inserting J_k in p_k:

$$p_k(t) = \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2(u_k(t) - u_k''(t)) + 2\lambda_0\gamma^2\beta_k U(t)$$
$$+ 2r\lambda_1\sigma_k^2 u_k(t) + \lambda_1(1-r^2)\sigma_k^2 J_k(t) + \lambda_1\gamma^2\beta_k J(t) + U_0\beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i\gamma^2}{2}.$$

After differentiation, we get

$$\begin{aligned} p_k''(t) = {} & r^2\alpha U_0(1-r^2)e^{-rt} + 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] + 2\lambda_0\gamma^2\beta U''(t) \\ & + 2r\lambda_1\sigma_k^2 u_k''(t) + \lambda_1(1-r^2)\sigma_k^2\left[-2ru_k(t) + r^2 J_k(t)\right] \\ & + \lambda_1\gamma^2\beta_k\left[-2rU(t) + r^2 J(t)\right] + r^2 U_0\beta_k \frac{2e^{-rt}}{r} \sum_{i=1}^{K_0} \lambda_{r,i} \frac{\beta_i\gamma^2}{2}. \end{aligned}$$

Finally, let

$$q_k(t) = p_k''(t) - r^2 p_k(t).$$

We have

$$\begin{aligned} q_k(t) = {} & 2\lambda_0\sigma_k^2\left[u_k''(t) - u_k''''(t)\right] - r^2 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] \\ & + 2\lambda_0\gamma^2\beta_k U''(t) - 2r^2\lambda_0\gamma^2\beta_k U(t) \\ & + 2r\lambda_1\sigma_k^2 u_k''(t) - 2r^3\lambda_1\sigma_k^2 u_k(t) \\ & - 2r\lambda_1\sigma_k^2(1-r^2)u_k(t) \\ & - 2r\lambda_1\gamma^2\beta_k U(t). \end{aligned} \tag{OA.81}$$

We must have $q_k(t) = 0$ for all k and all t, and this defines a differential equation that u_k must satisfy.

Now, let k denote the index of a nonexclusive signal. We have:

$$\begin{aligned}L_k(t) - L_k''(t) = \alpha_k U_0(1-r^2)e^{-rt} &+ 2\lambda_0\sigma_k^2\left[u_k(t) - u_k''(t)\right] + 2\lambda_0\gamma^2\beta_k U(t)\\ &+ \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k(t+\tau) + u_k(t-\tau)\right]\mathrm{d}\tau\\ &- \lambda_1\sigma_k^2\int_0^\infty e^{-r\tau}\left[u_k''(t+\tau) + u_k''(t-\tau)\right]\mathrm{d}\tau\\ &- \lambda_1\sigma_k^2\left(-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right) + \lambda_1\gamma^2\beta_k\int_0^\infty e^{-r\tau}U(t+\tau)\,\mathrm{d}\tau\\ &+ U_0\beta_k\frac{2e^{-rt}}{r}\sum_{i=1}^{K_0}\lambda_{r,i}\frac{\beta_i\gamma^2}{2} + (1-r^2)\lambda_{r,k}U_0\sigma_k^2\frac{e^{-rt}}{r}.\end{aligned}$$

Compared to the case of exclusive signals, we note the presence of one additional term in the expression for $L_k(t) - L_k''(t)$. However, this term being a constant factor of e^{-rt}, the chain of transformations we used for the case of exclusive signals continues to yield the same equation (OA.81). Hence, the differential equation obtained for u_k, by setting $q_k = 0$ when k denotes a nonexclusive signal, is the same as when k denotes an exclusive signal.

Thus, multiplying (OA.81) by $\frac{\beta_k}{2\sigma_k^2}$ and summing over k's, we get

$$\begin{aligned}\sum_{k=1}^K \frac{\beta_k}{2\sigma_k^2}q_k(t) = \lambda_0(U''(t) - U''''(t)) &- r^2\lambda_0(U(t) - U''(t)) + \lambda_0\gamma^2 m_\beta U''(t)\\ &- \lambda_0 r^2\gamma^2 m_\beta U(t) + r\lambda_1 U''(t) - r\lambda_1 U(t) - r\lambda_1\gamma^2 m_\beta U(t).\end{aligned}$$

As U is bounded, we can discard the positive roots. We conclude that U has the form

$$U(t) = C_1 e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t} + C_2 e^{-\kappa t}, \tag{OA.82}$$

for some constants C_1 and C_2.

Since $\sum_k \frac{\beta_k}{2\sigma_k^2}q_k = 0$, U is the solution of a homogeneous linear differential equation, whose characteristic polynomial has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$ and $\pm\sqrt{r(r+\lambda_1/\lambda_0)}$ (recall our assumption that $\lambda_1/\lambda_0 > -r$).

Next, let us fix an arbitrary pair (i,j) with $i \neq j$, define $\zeta_{ij}(t) = 2(\beta_i\sigma_i^2 u_j(t) - \beta_j\sigma_j^2 u_i(t))$. We have

$$\beta_i p_j(t) - \beta_j p_i(t) = \lambda_0(\zeta_{ij}''(t) - \zeta_{ij}''''(t)) - r^2\lambda_0(\zeta_{ij}(t) - \zeta_{ij}''(t)) + r\lambda_1\zeta_{ij}(t)'' - r\lambda_1\zeta_{ij}(t),$$

and since $\beta_i p_j - \beta_j p_i = 0$ must hold, we obtain a homogeneous linear differential equation that ζ_{ij} must satisfy. The roots of the characteristic polynomial are ± 1 and $\pm\sqrt{r(r+\lambda_1/\lambda_0)}$.

As ζ_{ij} is bounded, we can discard the positive roots. We conclude that ζ_{ij} has the form

$$\zeta_{ij}(t) = C_1' e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t} + C_2 e^{-\kappa t} + C_2' e^{-t}, \tag{OA.83}$$

for some constants C_1' and C_2'.

Putting together (OA.82) and (OA.83), it holds that

$$u_k(t) = D_{1,k} e^{-\sqrt{r(r+\lambda_1/\lambda_0)}t} + D_{2,k} e^{-\kappa t} + D_{3,k} e^{-t}, \tag{OA.84}$$

for some constants $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$. We can anticipate that $D_{3,k} = 0$, because U does not include a term e^{-t}.

Determination of the Constants

Let

$$\delta = \sqrt{r\left(r + \frac{\lambda_1}{\lambda_0}\right)}.$$

We plug the general form of u_k from (OA.84) into the expression for L_k. For both exclusive and nonexclusive signals S_k, L_k can be written in the form of a sum of three exponential terms, namely

$$L_k = L_{1,k} e^{-\delta t} + L_{2,k} e^{-\kappa t} + L_{3,k} e^{-t} + L_{4,k} e^{-rt},$$

where the constant factors $L_{1,k}$, $L_{2,k}$, and $L_{3,k}$ depend on the primitives of the model, as well as on the constants $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$. Asserting that $L_k = 0$ is equivalent to $L_{1,k} = L_{2,k} = L_{3,k} = 0$.

We solve for the constant factors $D_{1,k}$, $D_{2,k}$ and $D_{3,k}$, $k = 1, \ldots, K$, as well as the variables $\lambda_0, \lambda_1, \lambda_{r,k}$, using both the first-order condition that $L_k = 0$, and also the constraints $G_1(\mathbf{u}, \tau) = e^{-\tau}$ and $H_k(\mathbf{u}, \tau) = \beta_k(1 - e^{-\tau})$.

First, we note that for both exclusive and nonexclusive signals S_k, the term $L_{4,k}$ is

$$\frac{2\left(\lambda_0\left(r^2 - \kappa^2\right) + \lambda_1 r\right)\left(\gamma^2 \beta_k \sum_{i=1^K} \beta_i D_{2,i} - \left(\kappa^2 - 1\right)\sigma_k^2 D_{2,k}\right)}{(\kappa - 1)(\kappa + 1)(\kappa - r)(\kappa + r)}.$$

By assumption, we have

$$\left(\lambda_0\left(r^2 - \kappa^2\right) + \lambda_1 r\right) \neq 0,$$

which, together with $L_{4,k} = 0$, implies that

$$\gamma^2 \beta_k \sum_{i=1^K} \beta_i D_{2,i} - \left(\kappa^2 - 1\right)\sigma_k^2 D_{2,k} = 0,$$

so that, for every k,

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}. \tag{OA.85}$$

Second, we have, for both exclusive and nonexclusive signals S_k,

$$\begin{aligned}
L_{3,k} = {} & \frac{\gamma^2 U_0 \beta_k}{2(r-1)r} \sum_{i=1}^{K_0} \beta_i \lambda_{2,i} + \beta_k \sum_{i=1}^{K_0} \zeta_i \lambda_{2,i} \\
& + \frac{2\sigma_k^2 \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{r^2-1} D_{3,k} + \frac{\gamma^2 \beta_k \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{(\tau-1)\left(r^2-1\right)} \sum_{i=1}^{K} \beta_i D_{1,k} \\
& + \frac{\gamma^2 \beta_k \left(\lambda_0 \left(r^2-1\right) + \lambda_1 r\right)}{(\kappa-1)\left(r^2-1\right) \sum_{i=1}^{K} \beta_i D_{2,k}} + V_0 \beta_k,
\end{aligned}$$

for some expression ζ_i (whose expression is lengthy and therefore omitted). Thus, as

$$\lambda_0 \left(r^2-1\right) + \lambda_1 r \neq 0,$$

solving for $D_{3,k}$ in the equation $L_{3,k} = 0$ yields that $D_{3,k}$ is proportional to β_k/σ_k^2. Moreover, the term e^{-t} vanishes in (OA.82), which yields the equality

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0,$$

which, in turn, implies $D_{3,k} = 0$.

We have thus identified two factors. Therefore, we may already write

$$u_k(t) = D_{1,k} e^{-\tau t} + a\frac{\beta_k}{\sigma_k^2} e^{-\kappa t}.$$

Using this simplified expression for u_k, we also get

$$\begin{aligned}
\zeta_k = {} & \frac{\gamma^2 \beta_k (\tau + r + 2)}{2(\tau+1)r(r+1)(\tau+r)} \sum_{i=1}^{K_0} \beta_i D_{1,i} + \frac{\sigma_k^2}{r(\tau+r)} D_{1,k} \\
& + \frac{a\beta_k \left(\gamma^2 m_\beta (\kappa + r + 2) + 2(\kappa+1)(r+1)\right)}{2(\kappa+1)r(r+1)(\kappa+r)}.
\end{aligned}$$

Given this simplification in the expression for u_k, we now get to the remaining equations that characterize the unique solution to the problem. These are the equations (OA.86a)–(OA.86e) below. The numerator of $L_{4,k}$ for an exclusive signal S_k must be zero, which yields the equation

$$0=-\gamma^2U_0\beta_k(r-\tau)(r-\kappa)\sum_{i=1}^{K_0}\beta_i\lambda_{2,i}-\lambda_1 r\left(r^2-1\right)\sigma_k^2(r-\kappa)D_{1,k} + r(r-\tau)\left(a\lambda_1\beta_k\left(\gamma^2 m_\beta-r^2+1\right)+\left(r^2-1\right)U_0\alpha_k(r-\kappa)\right) + \gamma^2\lambda_1 r\beta_k(r-\kappa). \tag{OA.86a}$$

The numerator of $L_{4,k}$ for a nonexclusive signal S_k must be zero, which yields the equation

$$0=-\gamma^2U_0\beta_k(r-\tau)(r-\kappa)\left(\sum_{i=1}^{K_0}\beta_i\lambda_{2,i}\right) + (r-\tau)\left(\left(r^2-1\right)U_0\sigma_k^2(r-\kappa)\lambda_{2,k}+a\lambda_1 r\beta_k\left(\gamma^2 m_\beta-r^2+1\right)+r\left(r^2-1\right)U_0\alpha_k(r-\kappa)\right) - \lambda_1 r\left(r^2-1\right)\sigma_k^2(r-\kappa)D_{1,k}+\gamma^2\lambda_1 r\beta_k(r-\kappa)\sum_{i=1}^{K}\beta_i D_{1,k}. \tag{OA.86b}$$

Because $L_{3,k}=0$, we must have

$$0=\frac{\gamma^2U_0\beta_k}{2(r-1)r}\sum_{i=1}^{K_0}\beta_i\lambda_{2,i}+\beta_k\sum_{i=1}^{K_0}\zeta_i\lambda_{2,i} + \frac{\beta_k\left(a\gamma^2 m_\beta\left(\lambda_0\left(r^2-1\right)+\lambda_1 r\right)+(\kappa-1)\left(r^2-1\right)V_0\right)}{(\kappa-1)\left(r^2-1\right)} + \frac{\gamma^2\beta_k\left(\lambda_0\left(r^2-1\right)+\lambda_1 r\right)}{(\tau-1)\left(r^2-1\right)}\sum_{i=1}^{K}\beta_i D_{1,k}. \tag{OA.86c}$$

The public constraint, *i.e.*, the equation $G_1(\mathbf{u},\tau)=e^{-\tau}$, also yields an equation that involves a sum of three exponential terms, and can be written as

$$G_1e^{-\delta\tau}+G_2e^{-\tau}=e^{-\tau},$$

where G_1 and G_2 are constant factors obtained by plugging the expression derived for u_k above into the definition of the function $G(\mathbf{u},\tau)$, and by simplifying the resulting expressions, using $\kappa^2=1+\gamma^2 m_\beta$.

We must have $G_1=0$, which yields the equation

$$0=\frac{a\left(\tau^2+\gamma^2\left(-m_\beta\right)-1\right)}{\left(\tau^2-1\right)(\tau+\kappa)}\sum_{k=1}^{K}\beta_i D_{1,i}+\frac{\gamma^2}{2\tau-2\tau^3}\left(\sum_{k=1}^{K}\beta_i D_{1,i}\right)^2+\frac{1}{2\tau}\sum_{k=1}^{K}\sigma_i^2 D_{1,i}^2. \tag{OA.86d}$$

We then incorporate the constraint of nonexclusivity, $H_k(\mathbf{u},\tau) = \beta_k(1-e^{-\tau})$, for every nonexclusive signal S_k. After plugging the expression obtained for u_k just obtained, the term $G_{2,k}(\mathbf{u},\tau)$ is expressed as a sum of three exponentials and a constant term:

$$H_k(\mathbf{u},\tau) = H_{k,1}e^{-\delta\tau} + H_{k,2}e^{-\tau} + H_{k,3}.$$

Because $H_{k,2} = 0$, it holds that, for every nonexclusive signal index k,

$$D_{1,k} = \frac{\gamma^2\beta_k}{(\tau^2-1)\sigma_k^2}\sum_{k=1}^{K}\beta_i D_{1,i}. \tag{OA.86e}$$

Finally, we must also have $G_2 = 1$, which yields

$$\frac{1}{2}\gamma^2 U_0\left(-\frac{\sum_{k=1}^{K}\beta_i D_{1,i}}{\tau-1} - \frac{am_\beta}{\kappa-1}\right) + 1 = 0, \tag{OA.86f}$$

where we note that

$$U_0 = \frac{am_\beta}{\kappa+1} + \frac{1}{\tau+1}\sum_{k=1}^{K}\beta_i D_{1,i}.$$

However, we will not use (OA.86f). As will be verified, equations (OA.86a)–(OA.86e) yield a unique candidate $\mathbf{u}$, up to the scalar constant a which is pinned down by (OA.86f), and this candidate satisfies the first-order condition, and the public and nonexclusivity constraints.

Throughout, we use the notation

$$\rho_\beta := \frac{m_\beta^n}{m_\beta}.$$

We describe $(c_k^n, c_k^e, \lambda_{r,k}, \lambda_0)$ as a function of λ_1 and δ; then, λ_1 as a function of δ; finally, we define δ as one of the positive roots of some polynomial.

First, we briefly sketch how to solve the system, then state the solution.

1. First, we solve (OA.86a), (OA.86b) for c_k, taking the sum $\sum_{k=0}^{K} c_k\beta_k$ as a parameter. We multiply each by β_k, and add them up, giving us an affine equation for $\sum_{k=0}^{K} c_k\beta_k$, which we solve for.

2. Using the solution for each c_k, we obtain an expression for each of them, which is a function of λ_1, λ_0 and $\sum_{k=0}^{K_0}\beta_k\lambda_{r,k}$.

3. We plug the formula for c_k, $k \le K_0$ in (OA.86c), which yields an equation for each $\lambda_{r,k}$ (as a function of λ_1, λ_0 and $\sum_{k=0}^{K_0}\beta_k\lambda_{r,k}$).

4. Taking $\sum_{k=0}^{K_0}\beta_k\lambda_{r,k}$ as a parameter, we solve for each $\lambda_{r,k}$, multiply by β_k, add them up, which gives us an affine equation for $\sum_{k=0}^{K_0}\beta_k\lambda_{r,k}$, which we solve for.

5. We plug back into each of our formulas for c_k and $\lambda_{r,k}$, and obtain a solution for each of them, as a function of λ_0, λ_1 and δ only. Using that $\lambda_0 = \frac{r\lambda_1}{\delta^2 - r^2}$, we may eliminate λ_0 altogether, and obtain the formulas for c_k and $\lambda_{r,k}$ given below.

We then turn to (OA.86d) and (OA.86e). Substituting for c_k and $\lambda_{r,k}$, (OA.86e) becomes a quadratic expression for λ_1 (and independent of λ_0), while (OA.86d) is quadratic in λ_1 and affine in λ_0. Using that $\lambda_0 = \frac{r\lambda_1}{\delta^2 - r^2}$, this becomes another expression that is quadratic in λ_1 and independent of λ_0. We may eliminate the quadratic term by taking the weighted average of these two quadratic expressions, thereby obtaining an equation that is affine in λ_1. Solving it gives us λ_1 as a function of δ only, given below. Using that $\lambda_0 = \frac{r\lambda_1}{\delta^2 - r^2}$, the formula for λ_0 follows. Plugging these two formulas into either quadratic expression gives us a condition that δ must satisfy, which turns out to be precisely $P(\delta)P^*(\delta) = 0$, with P as defined below, and P^* a product of polynomials of degree no larger than two that admit no real roots.

The above procedure yields the solution

$$c_k^n = \frac{C_{1,k}}{R_k}, \quad c_k^e = \frac{C_{2,k}}{R_k},$$

where k is the index of a nonexclusive or an exclusive signal, and $C_{1,k}, C_{2,k}$ and R_k are given by

$$\begin{aligned}
C_{1,k} &= \left(\kappa^2 - 1\right)\beta_k(\delta - r)\left(m_{\alpha\beta}\left(r^2 - 1 - \left(\kappa^2 - 1\right)\rho_\beta\right)\right. \\
&\qquad \left. + (\kappa + r)\left((\kappa + 1)\lambda_1\left(\rho_\beta - 1\right) + m^n_{\alpha\beta}(\kappa - r)\right)\right), \\
C_{2,k} &= (r - \delta)\left((\kappa+1)\beta_k\left((\delta-1)\left((r+1)m_{\alpha\beta}(\delta-\kappa+1-r) + (\kappa+r)\left((\delta+1)\lambda_1 + m^n_{\alpha\beta}(r-\delta)\right)\right)\right.\right. \\
&\qquad \left. + \left(\kappa^2 - 1\right)\rho_\beta\left((\kappa - 1)m_{\alpha\beta} - \lambda_1(\kappa + r)\right)\right) \\
&\qquad \left. + (\delta - \kappa)\alpha_k m_\beta(\kappa + r)\left((\kappa + 1)\rho_\beta(\delta + \kappa - r - 1) - (\delta - 1)(r + 1)\right)\right), \\
R_k &= (\kappa + 1)\sigma_k^2\left((1 - \delta)m_{\alpha\beta}(\delta - r)\left(r^2 - 1 - \left(\kappa^2 - 1\right)\rho_\beta\right)\right. \\
&\qquad \left. + (r - \kappa)(\kappa + r)\left(\lambda_1\left(\left(\kappa^2 - 1\right)\rho_\beta - \delta^2 + 1\right) + (\delta - 1)m^n_{\alpha\beta}(\delta - r)\right)\right).
\end{aligned}$$

The Lagrangian coefficients $\lambda_{r,k}$ are given by

$$\lambda_{r,k} = \frac{L_{2,k}}{R'_k},$$

where

$$\begin{aligned}
L_{2,k} &= r\left((\kappa + 1)\beta_k\left((r + 1)m_{\alpha\beta}(\delta - r) + (\kappa + r)\left(\lambda_1(\delta + \kappa) + m^n_{\alpha\beta}(r + 1 - \delta - \kappa)\right)\right)\right. \\
&\qquad \left. + \alpha_k m_\beta(\kappa + r)\left((\kappa + 1)\rho_\beta(\delta + \kappa - r - 1) + (1 - \delta)(r + 1)\right)\right), \\
R'_k &= \sigma_k^2 m_\beta(\kappa + r)\left((\delta - 1)(r + 1) - (\kappa + 1)\rho_\beta(\delta + \kappa - r - 1)\right).
\end{aligned}$$

Finally,

$$\lambda_0 = \frac{r\lambda_1}{\delta^2 - r^2}.$$

It is time to define λ_1 given δ, namely

$$\lambda_1 = \frac{rQ_1Q_2}{(\delta+r)(r+\kappa)(1+\kappa)\delta_1}, \tag{OA.87}$$

where

$$Q_1 = (\delta+1)\left(\delta^2 - r\right) - (\kappa-1)\rho_\beta\left(r(\delta+\kappa+1) - \delta^2\right),$$

and

$$\begin{aligned} Q_2 = &\left(\kappa^2-1\right)^2(1-\rho_\beta)\, m_{\alpha\beta}^2\left((\kappa+1)\rho_\beta(2\delta+\kappa-1)+(\delta-1)^2\right) \\ &+ 2\left(\kappa^2-1\right) m_{\alpha\beta}m_{\alpha\beta}^e\left(r^2-\kappa^2\right)\left((\kappa+1)\rho_\beta(2\delta+\kappa-1)+(\delta-1)^2\right) \\ &+ (\kappa+r)^2\left(m_\beta m_\alpha^e\left((\kappa+1)\rho_\beta(r+1-\delta-\kappa)+(\delta-1)(r+1)\right)^2\right. \\ &\qquad \left. + (\kappa+1)(m_{\alpha\beta}^e)^2\left((\kappa+1)\rho_\beta(r+1-\delta-\kappa)^2+(1-\delta)^2(\kappa-2r-1)\right)\right), \end{aligned}$$

and

$$\delta_1 = d_0 + d_1\delta + d_2\delta^2 + d_3\delta^3 + d_4\delta^4 + d_5\delta^5 + d_6\delta^6,$$

with

$$\begin{aligned} d_0 = &-r^2\left(\left(\kappa^2-1\right)\rho_\beta\right. \\ &\left.+1\right)\left((\kappa-1)(\rho_\beta-1)\, m_{\alpha\beta}\left(2(\kappa+1)+(\kappa+1)\rho_\beta\left(-\left(-2\kappa^2+r(r+2)+2\right)\right)+r(\kappa+r+2)\right)\right. \\ &\left.+ m_{\alpha\beta}^e(\kappa+r)\left((\kappa+1)\rho_\beta\left(2\kappa^2-2\kappa(r+1)+r\right)+\kappa(r+2)-r\right)\right), \\ d_1 = &\, r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa-1)(\rho_\beta-1)\, m_{\alpha\beta}\left((r+1)(\kappa+r)-4\kappa(\kappa+1)\rho_\beta\right)\right. \\ &\left.+ m_{\alpha\beta}^e(\kappa+r)\left(\rho_\beta(-\kappa(4\kappa+1)+3\kappa r+r+1)+(\kappa-1)(r+1)\right)\right), \\ d_2 = &\, r\left((\kappa-1)(\rho_\beta-1)\, m_{\alpha\beta}\left(\kappa\right.\right. \\ &+(\kappa+1)\rho_\beta\left(5\kappa-2\left(\kappa^2-1\right)\rho_\beta(-2\kappa+2r+1)+\kappa^2(r+1)+\kappa r(r+2)-r(r(r+3)+9)-4\right) \\ &\left.+ r(3\kappa+r(\kappa+r+3)+5)+2\right) + m_{\alpha\beta}^e(\kappa+r)\left(\kappa\right. \\ &+ \rho_\beta\left(\kappa(\kappa(\kappa+5)-2)+\left(\kappa^2-1\right)\rho_\beta\left(\kappa(4\kappa-1)+2r^2-(7\kappa+1)r-1\right)-(\kappa-3)r^2+\right. \\ &\left.\left.\left.\left(\kappa\left(\kappa^2+\kappa-10\right)-2\right)r-2\right)+(\kappa-1)r^2+3\kappa r+r+1\right)\right), \\ d_3 = &\, m_{\alpha\beta}^e(\kappa \\ &+r)\left(\rho_\beta\left(2(\kappa-1)+\kappa^2\left(-4r^2+r+1\right)+\kappa r\left(r^2+r+6\right)+\kappa^3(3r-1)-r(r(r+5)+2)\right)\right. \\ &\left.+ (\kappa-1)(-r-1)\left(r^2+1\right)+(\kappa-1)(\kappa+1)\rho_\beta^2(\kappa+r(5\kappa-4r-1)-1)\right) \\ &- (\kappa-1)(\rho_\beta-1)\, m_{\alpha\beta}\left((r+1)\left(r^2+1\right)(\kappa+r)\right. \\ &\left.- (\kappa+1)\rho_\beta\left(\kappa-\kappa r^2+4\left(\kappa^2-1\right)r\rho_\beta+\kappa^2(3r-1)+(r+5)r\right)\right), \end{aligned}$$

$$
\begin{aligned}
d_4 = &-m^e_{\alpha\beta}(\kappa+r)\left(-\kappa\right.\\
&+\rho_\beta\left(\kappa\left(\kappa^2+\kappa+2\right)+(3\kappa-1)r^2+(\kappa-1)(\kappa+1)\rho_\beta(\kappa-2r-1)-\left(4\kappa^2+\kappa+5\right)r-2\right)\\
&\qquad\left.+r(\kappa+\kappa r+r+3)+1\right)-(\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}\left(-\kappa\right.\\
&\left.+(\kappa+1)\rho_\beta\left(\left(\kappa^2-1\right)\rho_\beta+\kappa^2+\kappa-2r^2-3(\kappa+1)r+1\right)+r(\kappa+r(\kappa+r+3)+3)\right),\\
d_5 = &(\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}\left((r+1)(\kappa+r)-4\kappa(\kappa+1)\rho_\beta\right)\\
&+m^e_{\alpha\beta}(\kappa+r)\left(\rho_\beta(-\kappa(4\kappa+1)+3\kappa r+r+1)+(\kappa-1)(r+1)\right),\\
d_6 = &m^e_{\alpha\beta}(\kappa+r)\left(-(\kappa+1)\rho_\beta-\kappa+2r+1\right)-(\kappa-1)\left(\rho_\beta-1\right)m_{\alpha\beta}\left((\kappa+1)\rho_\beta+\kappa-r\right).
\end{aligned}
$$

Finally, we are left with pinning down δ. Namely, δ must be one of the roots (if any) of the polynomial P, defined as

$$
P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6,
$$

with

$$
\begin{aligned}
a_0 = &r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa+r)^2\left(m_\beta m^e_\alpha\left(\left(\kappa^2-1\right)\rho_\beta+1\right)+\left(\kappa^2-1\right)(m^e_{\alpha\beta})^2\right)\right.\\
&\left.-2(\kappa-1)\kappa m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)^2\right),\\
a_1 = &2r^2\left(\left(\kappa^2-1\right)\rho_\beta+1\right)\left((\kappa+r)^2\left(m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)(m^e_{\alpha\beta})^2\right)\right.\\
&\left.+(1-\kappa)m_{\alpha\beta}m^e_{\alpha\beta}(2\kappa+r+1)(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\right),\\
a_2 = &r\left(2(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r)\left((1-\kappa)\rho_\beta\left(r^2-2\kappa(\kappa+1)+r\right)+\kappa\left(r^2+r+2\right)+\kappa^2(r+2)\right.\right.\\
&\left.-r(r+1)\right)+(\kappa+r)^2\left(m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)\left((\kappa-1)\rho_\beta(r-2\kappa-2)+r-2\right)\right.\\
&\left.+(\kappa-1)(m^e_{\alpha\beta})^2\left((\kappa-1)\rho_\beta(r-2\kappa-2)-\kappa(r+2)+r-2\right)\right)\\
&\left.+(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1)\left(2\left(\kappa^2-1\right)\rho_\beta+r^2+\kappa(r+2)+r+2\right)\right),\\
a_3 = &r\left(2(\kappa+r)^2\left(m_\beta m^e_\alpha\left((1-\kappa)\rho_\beta\left((\kappa-1)\rho_\beta+\kappa+3\right)-2\right)-(m^e_{\alpha\beta})^2\left((\kappa-1)^2\rho_\beta+\kappa^2-1\right)\right)\right.\\
&+4(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r)\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+r+1)+1\right)\\
&\left.+2(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}\left(\left(\kappa^2-1\right)\rho_\beta+\kappa(\kappa+2)+r^2+2(\kappa+1)r+2\right)\right),\\
a_4 = &-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}\left(\left(\kappa^2-1\right)\rho_\beta+2\kappa+1\right)\\
&+(\kappa+r)^2\left((\kappa-1)(m^e_{\alpha\beta})^2\left((\kappa-1)\rho_\beta+2\right)-m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)\left(\rho_\beta(1-\kappa)+2r-1\right)\right)\\
&+2(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r)\left((\kappa-1)\rho_\beta(r-\kappa)+r-2\kappa\right),\\
a_5 = &2(\kappa+r)^2\left(m_\beta m^e_\alpha\left((\kappa-1)\rho_\beta+1\right)+(\kappa-1)(m^e_{\alpha\beta})^2\right)\\
&-2(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(2\kappa+r+1)(\kappa+r)-2(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}(\kappa+r+1),\\
a_6 = &m_\beta m^e_\alpha(\kappa+r)^2-2(\kappa-1)m_{\alpha\beta}m^e_{\alpha\beta}(\kappa+r)-(\kappa-1)^2\left(\rho_\beta-1\right)m^2_{\alpha\beta}.
\end{aligned}
$$

OA.11.2 Verification of Optimality

We verify that the candidate for optimal ratings just obtained is optimal. As in the other three cases, we use the auxiliary setting introduced in Section OA.5, but we redefine the principal's instantaneous payoff.

We introduce the extra state variables Λ, $\Lambda_1^r, \ldots, \Lambda_{K_0}^r$, with initial value

$$\Lambda_0 = 0,$$
$$\Lambda_{k,0}^r = \frac{1}{r}\int_{s\le 0} e^{-r(t-s)}\,\mathrm{d}S_{k,s},$$

and which evolve according to

$$\mathrm{d}\Lambda_0 = -r\Lambda_t\,\mathrm{d}t + Y_t\,\mathrm{d}t,$$
$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right].$$

We let the principal's instantaneous payoff function be defined as

$$H_t = c'(A_t) - \phi_1 Y_t(Y_t - \nu_t) \\ - \phi_2 Y_t\left(\frac{Y_t}{1+r} - \Lambda_t\right) + \sum_{k=1}^{K_0}\phi_{3,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)} - Y_t\Lambda_{k,t}^r\right),$$

where ϕ_1 is the fraction with numerator

$$m_{\alpha\beta}(r-\delta)\Big(\rho_{\alpha\beta}(\delta-1)(r+\kappa)((r^2-\delta)(\kappa-1)+r(1+3\delta+3\kappa+\delta\kappa)) \\ + \rho_\beta(\kappa^2-1)(\delta^2+r(4+3r+r^2-\delta^2)) - 4r(r+1)(\delta^2-1)\Big) \\ + \lambda_1(\kappa+1)(r+\kappa)\Big(2r(\delta^2-1)(\delta+1) - \rho_\beta(\kappa-1)((r^2+\delta^2)(\delta+\kappa)+2r(1+\delta-\delta^2+\kappa))\Big),$$

and denominator

$$(r+1)(r-\delta)(r+\delta)(\kappa-1)(r+\kappa)((r+1)(\delta-1)+\rho_\beta(1+r-\delta-\kappa)(\kappa+1)),$$

with, as in the first half of this proof, $\rho_\beta := m_\beta^n/m_\beta$ and $\rho_{\alpha\beta} := m_{\alpha\beta}^n/m_{\alpha\beta}$. Additionally,

$$\phi_2 := \frac{(r+1)(r^2-\delta^2)}{r+\delta^2}\phi_1,$$
$$\phi_{3,k} := r\frac{\alpha_k}{\sigma_k^2}$$

$$-\frac{r(1+\kappa)\Big(m_{\alpha\beta}(1+r)(-r+\delta)+(r+\kappa)(m^n_{\alpha\beta}(1+r-\delta-\kappa)+(\delta+\kappa)\lambda_1)\Big)}{(r+\kappa)(m_\beta(1+r)(\delta-1)+m^n_\beta(1+r-\delta-\kappa)(1+\kappa))}\frac{\beta_k}{\sigma_k^2},$$

where λ_1 is defined by Equation (OA.87) in the first half of this proof. In the remainder of this proof, we also use the following notation:

$$\begin{aligned}
\Phi_{3,\alpha} &\coloneqq \sum_{k=1}^{K_0} \alpha_k \phi_{3,k} \\
&= rm^n_\alpha - \frac{r(1+\kappa)\Big(m_{\alpha\beta}(1+r)(-r+\delta)+(r+\kappa)(m^n_{\alpha\beta}(1+r-\delta-\kappa)+(\delta+\kappa)\lambda_1)\Big)}{(r+\kappa)(m_\beta(1+r)(\delta-1)+m^n_\beta(1+r-\delta-\kappa)(1+\kappa))} m^n_{\alpha\beta}, \\
\Phi_{3,\beta} &\coloneqq \sum_{k=1}^{J_0} \beta_k \phi_{3,k} \\
&= rm^n_{\alpha\beta} - \frac{r(1+\kappa)\Big(m_{\alpha\beta}(1+r)(-r+\delta)+(r+\kappa)(m^n_{\alpha\beta}(1+r-\delta-\kappa)+(\delta+\kappa)\lambda_1)\Big)}{(r+\kappa)(m_\beta(1+r)(\delta-1)+m^n_\beta(1+r-\delta-\kappa)(1+\kappa))} m^n_\beta.
\end{aligned}$$

The value of Δ is defined as in Section B.3.1.

Compared to the benchmark confidential exclusive setting, the principal's payoff includes $2+K_0$ penalty terms. The penalty terms $\phi_1 Y_t(Y_t-\nu_t)$ and $\phi_2 Y_t\left(\frac{Y_t}{1+r}-\Lambda_t\right)$ play the same role as in the public exclusive case: they ensure that the optimal transfer of the principal remains close to a public market belief (however, the two penalty factors ϕ_1 and ϕ_2 differ from the public exclusive case). The penalty terms $-\phi_{3,k}\left(\frac{\beta_k(\kappa^2-1)}{2m_\beta r(1+r)}-Y_t\Lambda^r_{k,t}\right)$, one for each nonexclusive signal S_k, ensure that the optimal transfer is close to a market belief that incorporates the relevant information of the nonexclusive signals (compared to the confidential nonexclusive setting, only one state variable per signal is required).

The principal's problem is an optimal control problem whose state variables are: the agent's estimate of his ability ν, the agent's continuation transfer J, and the aforementioned state variables associated with the public and nonexclusive constraints. The state variables evolve as follows:

$$\begin{aligned}
\mathrm{d}J_t &= (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\frac{\xi_\beta}{m_\beta}\frac{\kappa-1}{1+r}\frac{\beta_k}{\sigma_k^2}+C_k\right)[\mathrm{d}S_{k,t}-(\alpha_k A_t+\beta_k\nu_t)\,\mathrm{d}t], \\
\mathrm{d}\nu_t &= -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}[\mathrm{d}S_{k,t}-\alpha_k A_t\,\mathrm{d}t], \\
\mathrm{d}\Lambda_t &= (-r\Lambda_t+Y_t)\,\mathrm{d}t, \\
\mathrm{d}\Lambda^r_{k,t} &= -r\Lambda^r_{k,t}\,\mathrm{d}t + \frac{1}{r}[\mathrm{d}S_{k,t}-\alpha_k A_t\,\mathrm{d}t], \quad k=1,\ldots,K_0.
\end{aligned}$$

As in previous cases, $\xi_\beta \coloneqq \sum_k \beta_k C_k$ and $C_k \coloneqq \int_{\tau \geq 0} e^{-r\tau} u_k(\tau)\,\mathrm{d}\tau$.

As in the proof of the baseline exclusive setting detailed in Section se:excluconf, the principal's problem reduces to finding a stationary linear contract (A, Y), along with processes $\widehat{C}_k$, $k = 1, \ldots, K$, such that, for all t, the principal maximizes

$$\mathbf{E}\left[\int_t^\infty \rho e^{-\rho(s-t)} H_s\,\mathrm{d}s \,\middle|\, \mathcal{R}_t\right]$$

subject to:

1. Incentive compatibility: $c'(A_t) = \widehat{\xi}_\alpha \coloneqq \sum_k \alpha_k \widehat{C}_k$.
2. The evolution of the agent's belief ν,
$$\mathrm{d}\nu_t = -\kappa\nu_t\,\mathrm{d}t + \frac{\kappa-1}{m_\beta}\sum_{k=1}^{K}\frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,s} - \alpha_k A_s\,\mathrm{d}s\right].$$
3. The evolution of the agent's continuation transfer J,
$$\mathrm{d}J_t = (rJ_t - Y_t)\,\mathrm{d}t + \sum_{k=1}^{K}\left(\widehat{\xi}_{\beta,t}\frac{\gamma^2}{(1+\kappa)(1+r)}\frac{\beta_k}{\sigma_k^2} + \widehat{C}_{k,t}\right)\left[\mathrm{d}S_{k,t} - (\alpha_k A_t + \beta_k\nu_t)\,\mathrm{d}t\right],$$
with $\widehat{\xi}_\beta \coloneqq \sum_k \beta_k \widehat{C}_k$.
4. The evolution of the state Λ,
$$\mathrm{d}\Lambda_t = (-r\Lambda_t + Y_t)\,\mathrm{d}t.$$
5. The evolution of the states Λ_k^r, for $k = 1, \ldots, K_0$,
$$\mathrm{d}\Lambda_{k,t}^r = -r\Lambda_{k,t}^r\,\mathrm{d}t + \frac{1}{r}\left[\mathrm{d}S_{k,t} - \alpha_k A_t\,\mathrm{d}t\right],\ \forall k = 1, \ldots, K_0.$$
6. The following transversality conditions
$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau} \mid \mathcal{R}_t] = 0,\ \text{and}$$
$$\lim_{\tau\to+\infty}\mathbf{E}[e^{-\rho\tau}J_{t+\tau}^2 \mid \mathcal{R}_t] = 0.$$

Using dynamic programming without imposing linearity or stationarity of the transfer processes, and assuming that the principal's value function V is jointly twice continuously differentiable, the HJB equation is:

$$\begin{aligned}
\rho V = \sup_{y, c_1, \ldots, c_K} & \rho \widehat{\xi}_\alpha - \rho \phi_1 y(y-\nu) - \rho \phi_2 y \left(\frac{y}{1+r} - \Lambda \right) \\
& + \rho \sum_{k=1}^{K_0} \phi_{3,k} \left(\frac{\beta_k (\kappa^2 - 1)}{2 m_\beta r (1+r)} - y \Lambda_k^r \right) + (rJ - y) V^J - \nu V^\nu + (-r\Lambda + y) V^\Lambda \\
& + \sum_{k=1}^{K_0} \left(-r \Lambda_k^r + \frac{\beta_k}{r} \nu \right) V_k^r + \frac{V^{JJ}}{2} \sum_{k=1}^{K} \sigma_k^2 \left(\frac{\widehat{\xi}_\beta}{m_\beta} \frac{\kappa - 1}{1+r} \frac{\beta_k}{\sigma_k^2} + C_k \right)^2 \\
& + \frac{V^{J\nu} \widehat{\xi}_\beta (\kappa - 1)(\kappa + r)}{m_\beta (1+r)} + \frac{V^{\nu\nu} (\kappa - 1)^2}{2 m_\beta} + \sum_{k=1}^{K_0} \frac{V_k^{Jr}}{r} \left(\frac{\widehat{\xi}_\beta (\kappa - 1) \beta_k}{m_\beta (1+r)} + \sigma_k^2 c_k \right) \\
& + \sum_{k=1}^{K_0} \frac{V_k^{\nu r}}{r} \frac{(\kappa - 1)\beta_k}{m_\beta} + \sum_{k=1}^{K_0} \frac{V_{kk}^{rr}}{2r^2} \sigma_k^2 .
\end{aligned}$$

As in Section OA.9 which proves the optimal rating of the nonexclusive confidential case, we have used a short superscript/subscript notation for the (partial) derivatives of V, where we use superscripts to denote the variables $(\nu, J, \Lambda, \Lambda^r)$, and subscripts to denote the index of the variable Λ^r. Besides, we abuse notation by using $\widehat{\xi}_\alpha$ and $\widehat{\xi}_\beta$ to denote $\sum_k \alpha_k c_k$ and $\sum_k \beta_k c_k$, respectively.

We conjecture a quadratic value function V of the form

$$\begin{aligned}
V(J, \nu, \Lambda, \Lambda^r) = & a_0 + a^J J + a^\nu \nu + a^\Lambda \Lambda + a^{J\nu} J \nu + a^{J\Lambda} J \Lambda + a^{\nu\Lambda} \nu \Lambda \\
& + a^{JJ} J^2 + a^{\nu\nu} \nu^2 + a^{\Lambda\Lambda} \Lambda^2 + \sum_{k=1}^{K_0} \left(a_k^r \Lambda_k^r + a_k^{Jr} J \Lambda_k^r + a_k^{\nu r} \nu \Lambda_k^r + a_k^{\Lambda r} \Lambda \Lambda_k^r \right) \\
& + \sum_{1 \le k \le j \le K_0} a_{kj}^{rr} \Lambda_k^r \Lambda_j^r .
\end{aligned}$$

After substituting this quadratic expression into the HJB equation, we solve for the optimal control variables $y, c_1, \ldots, c_K$. The right-hand side of the resulting equation is a sum of two quadratic functions, one in y, the other in $(c_1, \ldots, c_K)$. These quadratic functions are strictly concave when the following second-order conditions are satisfied:

$$\phi_1 + \frac{\phi_2}{1+r} > 0 \quad \text{and} \quad a^{JJ} < 0. \tag{OA.88}$$

We will later verify that, if the principal's discount rate is low enough, the second-order conditions hold. If the second-order conditions hold, then the first-order conditions yield the values of the optimal control variables, which in turn allow us to identify the coefficients of the quadratic value function.

As for the public exclusive case studied in Section OA.8, there are two sets of coefficients that satisfy the equations of the first-order conditions. However, only one solution gives a state J that satisfies the transversality conditions. It is therefore the solution that we keep.

It is helpful to introduce the following variables:

$$\begin{aligned}
\bar{\phi} &:= (1+r)\phi_1 + \phi_2, \\
\bar{\rho} &:= \sqrt{\bar{\phi}(2r+\rho)((1+r)(2r+\rho)\phi_1 - (2-\rho)\phi_2)}, \\
&= (1+r)\sqrt{\frac{\bar{\phi}\phi_1(2r+\rho)(2\delta^2+r\rho)}{r+\delta^2}}, \\
\bar{\varphi}_\pm &:= \bar{\rho} + (2r \pm \rho)\bar{\phi}.
\end{aligned}$$

We then get:

$$a^J = a^\nu = a^\Lambda = a_k^r = 0, \quad \forall k = 1, \ldots, K_0,$$

and

$$\begin{aligned}
a^{JJ} &= -\frac{(2r-\rho)\rho}{8r^2(r+1)}\Big(\bar{\varphi}_-(2r-\rho) + 4\phi_1 r\rho(r+1) + \phi_2(-2r-2r^2-\rho+3r\rho)\Big), \\
a^{J\nu} &= \frac{(2r-\rho)\rho\bar{\varphi}_-\Big(2\phi_1\bar{\varphi}_+ r^2(1+r+\rho) + \Phi_{3,\beta}(2\bar{\rho}(r+1) - \bar{\varphi}_+(2+2r+\rho))\Big)}{4r^3(r+1)(\bar{\rho}+2\bar{\phi}+\rho\bar{\phi})\bar{\varphi}_+}, \\
a^{J\Lambda} &= \frac{(2r-\rho)\rho(\bar{\rho}\rho + (2r+\rho)(\phi_2 + r\phi_2 - \rho\bar{\phi}))}{4r^2(r+1)}, \\
a^{\nu\nu} &= \left[16(r^2-(1+\rho)^2)^2\phi_1^2 + \frac{8d_1(1-r+\rho)\phi_1\Phi_{3,\beta}}{r^3\bar{\varphi}_+} + \frac{d_2\Phi_{3,\beta}^2}{r^6\bar{\varphi}_+}\right] \\
&\quad \times \frac{\rho\bar{\phi}}{16(r+1)(2+\rho)(\bar{\rho}+2\bar{\phi}+\rho\bar{\phi})^2}, \\
a^{\nu\Lambda} &= \frac{\rho^2(2\bar{\rho}-\bar{\varphi}_+)\Big(-2\phi_1\bar{\varphi}_+ r^2(1-r+\rho) + \Phi_{3,\beta}(\bar{\varphi}_+(-2r+\rho) + 2\bar{\phi}\rho(r+1))\Big)}{4r^3(r+1)(\bar{\rho}+2\bar{\phi}+\rho\bar{\phi})\bar{\varphi}_+}, \\
a^{\Lambda\Lambda} &= \frac{\rho^3(-\bar{\rho} + (1+r)(2r+\rho)\phi_1 - (1-r-\rho)\phi_2)}{8r^2(1+r)},
\end{aligned}$$

where the constants for the term $a^{\nu\nu}$ are given by

$$\begin{aligned}
d_1 &= (1+r+\rho)\Big(\bar{\varphi}_-(4r^2-\rho^2)(1+r+\rho) + \rho^2(1-r+\rho)(2\bar{\rho}-\bar{\varphi}_+)\Big) \\
&\qquad + \frac{4r(r+1)\rho(\bar{\rho}-\bar{\varphi}_+)}{\bar{\rho}+2\bar{\phi}+\rho\bar{\phi}}\Big(-\phi_2(r+1)(2r+\rho) + 2\bar{\phi}(r-1)^2 + \bar{\varphi}_+(2+\rho)\Big),
\end{aligned}$$

$$
\begin{aligned}
d_2 = &\Big(2\bar{\rho}\rho^2(1-r+\rho) + \bar{\varphi}_-(4r^2-\rho^2)(1+r+\rho) - \bar{\varphi}_+\rho^2(1-r+\rho)\Big)^2 \\
&- \frac{16r^2(r+1)\rho(\bar{\rho}-\bar{\varphi}_+)}{\bar{\rho}+2\bar{\phi}+\rho\bar{\phi}}\Big(-\phi_2(r+1)(2r+\rho) + 2\bar{\phi}(r-1)^2 + \bar{\varphi}_+(2+\rho)\Big) \\
&\times (2\rho\bar{\phi} + 2r\rho\bar{\phi} - 2r\bar{\varphi}_+ + \rho\bar{\varphi}_+).
\end{aligned}
$$

Next, for $1 \le k \le K_0$,

$$
\begin{aligned}
a_k^{Jr} &= -\frac{\rho(4r^2-\rho^2)\bar{\varphi}_-\phi_{3,k}}{4r^2\bar{\varphi}_+}, \\
a_k^{\nu r} &= \frac{\rho^2(\bar{\rho}-\bar{\varphi}_+)\Big(\Phi_{3,\beta}\big(-2\bar{\phi}\rho(1-r+\rho) + \bar{\varphi}_-(2r-\rho)\big) + 2\phi_1\bar{\varphi}_+r^2(1-r+\rho)\Big)\phi_{3,k}}{2r^3(\bar{\rho}+2\bar{\phi}+\rho\bar{\phi})\bar{\varphi}_+^2}, \\
a^{\Lambda r} &= -\frac{\rho^3(\bar{\varphi}_+-2\bar{\rho})\phi_{3,k}}{4r^2\bar{\varphi}_+}.
\end{aligned}
$$

Finally,

$$
a_{kj}^{rr} = \begin{cases} \frac{(1+r)\rho^3(\bar{\varphi}_+-\bar{\rho})\phi_{3,k}\phi_{3,j}}{2r^2\bar{\varphi}_+^2} & \text{if } k < j, \\ \frac{(1+r)\rho^3(\bar{\varphi}_+-\bar{\rho})\phi_{3,k}^2}{4r^2\bar{\varphi}_+^2} & \text{if } k = j. \end{cases}
$$

The constant term a_0 is unwieldy and irrelevant for the sequel. Therefore, it is omitted.

Having derived the expressions for a^{JJ}, let us briefly return to the second-order conditions. Let us assume $\phi_1 > 0$, which can be verified. The first of the second-order conditions given by (OA.88) is

$$
\phi_1 + \frac{\phi_2}{1+r} = \frac{\bar{\phi}}{1+r} = \frac{r(r+1)}{r+\delta^2}\phi_1 > 0,
$$

and, as $\rho \to 0$,

$$
\frac{a^{JJ}}{\rho} \to -\frac{(r+\delta)^2}{2r}\left(\phi_1 + \frac{\phi_2}{1+r}\right) < 0.
$$

Therefore, for ρ close enough to zero, the second-order conditions are satisfied.

The controls are then expressed as follows. If $k > K_0$, *i.e.*, if S_k is an exclusive signal, then

$$
c_k(J,\nu,\Lambda,\Lambda^r) = \frac{1}{\zeta}\left(\frac{4r^2(r+1)\alpha_k}{(2r-\rho)\sigma_k^2} + \frac{2d_\beta(r+1)(\kappa-1)\beta_k}{m_\beta(r+\kappa)^2(2r-\rho)\sigma_k^2} - \frac{(r+1)(2r+\rho)\bar{\varphi}_-\phi_{3,k}}{r\bar{\varphi}_+}\right),
$$

while if $k \leq K_0$, *i.e.*, if S_k is a nonexclusive signal,

$$c_k(J, \nu, \Lambda, \Lambda^r) = \frac{1}{\zeta}\left(\frac{4r^2(r+1)\alpha_k}{(2r-\rho)\sigma_k^2} + \frac{2d_\beta(r+1)(\kappa-1)\beta_k}{m_\beta(r+\kappa)^2(2r-\rho)\sigma_k^2}\right),$$

where

$$\begin{aligned} \zeta &:= \bar{\varphi}_-(2r-\rho) + (2r+\rho)(r\phi_1 + r^2\phi_1 - \phi_2) + \bar{\phi}(-2r^2 + 3r\rho), \\ d_\beta &:= -2m_{\alpha\beta}r^2(1+2r+\kappa) + \frac{(r+\kappa)(2r-\rho)\bar{\varphi}_-(\Phi_{3,\beta}(2\bar{\rho} - \bar{\varphi}_+) + \phi_1\bar{\varphi}_+ r(1+r+\rho))}{(\bar{\rho} + 2\bar{\phi} + \rho\bar{\phi})\bar{\varphi}_+}. \end{aligned}$$

Finally,

$$y(J, \nu, \Lambda, \Lambda^r) = \begin{bmatrix} b^J \\ b^\nu \\ b^\Lambda \\ \boldsymbol{b}^r \end{bmatrix} \cdot \begin{bmatrix} J \\ \nu \\ \Lambda \\ \boldsymbol{\Lambda}^r \end{bmatrix},$$

where

$$\begin{aligned} b^J &= \frac{(2r-\rho)\bar{\varphi}_-}{4r\bar{\phi}}, \\ b^\nu &= \frac{(-r^2 + (1+\rho)^2)\phi_1}{\bar{\rho} + 2\bar{\phi} + \rho\bar{\phi}} + \frac{2\bar{\rho}\rho(r+1) + \bar{\varphi}_+(2r + 2r^2 - 2\rho - \rho^2)}{2r^2(\bar{\rho} + 2\bar{\phi} + \rho\bar{\phi})\bar{\varphi}_+}\Phi_{3,\beta}, \\ b^\Lambda &= \frac{\rho(\bar{\varphi}_+ - 2\bar{\rho})}{4r\bar{\phi}}, \\ b^r &= -\frac{(1+r)\rho(2r+\rho)}{2r\bar{\varphi}_+}\boldsymbol{\phi}_3. \end{aligned}$$

In the above equations, $\boldsymbol{\phi}_3 = (\phi_{3,1}, \ldots, \phi_{3,K_0})$.

Plugging these controls back into the equations of evolution for the state variables, we obtain a $(3+K_0)$-dimensional stochastic differential equation:

$$\mathrm{d}\begin{bmatrix} J_t \\ \nu_t \\ L_t \\ \boldsymbol{\Lambda}_t^r \end{bmatrix} = M \begin{bmatrix} J_t \\ \nu_t \\ L_t \\ \boldsymbol{\Lambda}_t^r \end{bmatrix} + \sum_{k=1}^{K} \begin{bmatrix} \frac{\widehat{\xi}_\beta(\kappa-1)\beta_k}{m_\beta(r+1)\sigma_k^2} + \widehat{C}_k \\ \frac{\kappa-1}{m_\beta}\frac{\beta_k}{\sigma_k^2} \\ 0 \\ \mathbf{e}_k/r \end{bmatrix} \left[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t\right],$$

where

$$M = \begin{bmatrix} r - b^J & -b^\nu - \widehat{\xi}_\beta \frac{\kappa+r}{r+1} & -b^\Lambda & -(\mathbf{b}^r)^\intercal \\ 0 & -\kappa & 0 & 0 \\ b^J & b^\nu & b^\Lambda - r & (\mathbf{b}^r)^\intercal \\ 0 & 0 & 0 & -r\mathbf{I}_{K_0} \end{bmatrix}$$

In these equations, $\boldsymbol{I}_{K_0}$ denotes the $K_0 \times K_0$ identity matrix, and $\mathbf{e}_k \in \mathbf{R}^{K_0}$ is the vector with one in the k-th component and zeros elsewhere. The matrix M has four eigenvalues, $-\delta_\rho, -\kappa, -r, -(r-\rho)$, where

$$\delta_\rho := \frac{\bar{\rho} - \rho\bar{\phi}}{2\bar{\phi}}.$$

These eigenvalues are distinct and negative for sufficiently small $\rho > 0$ if the second-order conditions (OA.88) are satisfied. Therefore, we can write

$$\begin{bmatrix} J_t \\ \nu_t \\ \Lambda_t \\ \mathbf{\Lambda}^r \end{bmatrix} = \sum_{k=1}^{K} \int_{s \le t} \left(\mathbf{f}_k^\delta e^{-\delta_\rho (t-s)} + \mathbf{f}_k^\kappa e^{-\kappa(t-s)} + \mathbf{f}_k^r e^{-r(t-s)} + \mathbf{f}_k^\rho e^{-\rho(t-s)} \right) [\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t],$$

for some $(3+K_0)$-vectors $\mathbf{f}_k^\delta, \mathbf{f}_k^\kappa, \mathbf{f}_k^r, \mathbf{f}_k^\rho$ $(k = 1, \dots, K_0)$ that can be expressed in closed form as a function of the parameters of the model (as in the other cases, the expression for $\rho > 0$ is lengthy and thus omitted). It follows that

$$Y_t = \sum_{k=1}^{K} \int_{s \le t} u_k(t-s) \,[\mathrm{d}S_{k,t} - \alpha_k A_t \,\mathrm{d}t],$$

with

$$u_k(\tau) = F_k^\delta e^{-\delta_\rho \tau} + F_k^\kappa e^{-\kappa\tau} + F_k^r e^{-r\tau} + F_k^\rho e^{-(r-\rho)\tau}.$$

In the limit as $\rho \to 0$, each factor converges, the first exponential to an exponential with rate $-\delta$ and the last two to a single exponential with rate $-r$ and factor $F_k^r + F_k^\rho$. For nonexclusive signals S_k, $k \le K_0$, we have

$$\begin{aligned} F_k^\delta &\to ad^n \frac{\beta_k}{\sigma_k^2}, \\ F_k^\kappa &\to a \frac{\beta_k}{\sigma_k^2}, \\ F_k^r + F_k^\rho &\to 0, \end{aligned}$$

while for the exclusive signals $S_k, k > K_0$,

$$F_k^\delta \to a\left(c^e \frac{\beta_k}{\sigma_k^2} + d^e \frac{\alpha_k}{\sigma_k^2}\right),$$
$$F_k^\kappa \to a\frac{\beta_k}{\sigma_k^2},$$
$$F_k^r + F_k^\rho \to 0,$$

where d^n, c^e, d^e and $\delta > 0$ are defined as in Part I of this proof.

The scaling factor is

$$a = -\frac{(r+1)(r-\delta)(r-\kappa)(\kappa-1)\Phi_{3,\beta}}{2m_\beta r(\delta+1)(r+\delta)(\delta-\kappa)\bar{\phi}} - \frac{m_{\alpha\beta} r(r+1)^2(\kappa-1)}{m_\beta(r+\delta)(\delta-\kappa)(r+\kappa)\bar{\phi}} + \frac{(r+\delta^2)(r-\kappa)(\kappa-1)}{2m_\beta r(1+\delta)(\delta-\kappa)}.$$

The limit of $\widehat{\xi}_\alpha$ as $\rho \to 0$ is

$$c'(A) = -\frac{(r+1)\Phi_{3,\alpha}}{(r+\delta)^2\bar{\phi}} - \frac{m_{\alpha\beta}(r+1)(r-\delta)(\kappa-1)\Phi_{3,\beta}}{m_\beta(\delta+1)(r+\delta)^2(r+\kappa)\bar{\phi}} + \frac{r(r+1)(m_\alpha m_\beta(r+\kappa)^2 - m_{\alpha\beta}^2(\kappa-1)(1+2r+\kappa))}{m_\beta(r+\delta)^2(r+\kappa)^2\bar{\phi}} + \frac{m_{\alpha\beta}(r+\delta^2)(\kappa-1)}{m_\beta(1+\delta)(r+\delta)(r+\kappa)}.$$

As in the other cases, we observe that, as $\rho \to 0$, the value obtained for $c'(A)$ corresponds to the conjectured optimum of the original model, and the optimal transfer Y corresponds to the conjectured optimal market belief of the optimal rating of the original model.

Back to the Original Model

We now conclude the verification. The arguments are similar to those of the public exclusive setting explained in Section OA.8. Therefore, we skip the details.

Let (A^*, Y^*) be the incentive-compatible contract defined by Y^* as the market belief of the conjectured optimal rating of the original setting, given by the linear filter described at the end of Part I, and A_t^* the associated conjectured optimal action.

Let $\widehat{\mathcal{M}}$ be a confidential information structure with nonexclusive signals S_k, $k \leq K_0$, associated with market belief $\widehat{Y}$ and stationary action $\widehat{A}$; $(\widehat{A}, \widehat{Y})$ is then a well-defined incentive-compatible stationary linear contract. We want to show that $c'(A^*) \geq c'(\widehat{A})$.

Let $(A^{(\rho)}, Y^{(\rho)})$ be the optimal incentive-compatible stationary linear contract defined as the optimal solution above, as a function of the discount rate of the principal ρ, with $V^{(\rho)}$ the corresponding principal's expected payoff.

Let V^* be the expected payoff for contract (A^*, Y^*), and $\widehat{V}$ be the expected payoff for contract $(\widehat{A}, \widehat{Y})$. As $\rho \to 0$, by a direct extension of the argument used in Section OA.8, $\rho V^* \to c'(A^*)$ and $\rho \widehat{V} \to c'(\widehat{A})$.

For every $\rho \in (0, r)$, the inequalities $\rho V^{(\rho)} \geq \rho \widehat{V} = c'(\widehat{A})$ must hold. However, as $\rho \to 0$, it holds that $c'(A^{(\rho)}) \to c'(A^*)$, and the linear filter of $Y^{(\rho)}$ converges pointwise to the linear filter of Y^*. By an argument similar to the one in the exclusive public setting of Section OA.8, this implies that $\rho V^{(\rho)} - \rho V^* \to 0$ as $\rho \to 0$, so $\rho V^{(\rho)} \to c'(A^*)$ and hence $c'(A^*) \geq c'(\widehat{A})$.

OA.12 Proof of Proposition 5.5

For a rating process Y with linear filter $\{u_k\}_k$, let

$$V_\ell(t) = \sum_{k=1}^{K} \alpha_{k,\ell} u_k(t),$$

for $\ell = 1, \ldots, L$. The extension of Equation (32) of Lemma B.3 to the multiaction setting is immediate: the equilibrium actions $A_1, \ldots, A_L$ are pinned down by the first-order conditions

$$c'(A_\ell) = \frac{\mathbf{Cov}\left[\theta_t, Y_t\right]}{\mathbf{Var}\left[Y_t\right]} \int_0^\infty e^{-rt} V_\ell(t) \, \mathrm{d}t, \qquad \forall \ell = 1, \ldots, L. \tag{OA.89}$$

As in the baseline model, the equilibrium action is constant.

The rater seeks to maximize the discounted expected output, which is equivalent to maximizing the drift of the output process,

$$\sum_{\ell=1}^{L} \alpha_{1,\ell} A_\ell,$$

over linear filters $\{u_k\}_k$ that define the rating process Y by

$$Y_t = \sum_{k=1}^{K} \int_{s \leq t} u_k(t-s) \, \mathrm{d}S_{k,s},$$

subject to (OA.89). With quadratic costs, using without loss a normalized factor $c = 1$, this optimization problem reduces to maximizing

$$\frac{\mathbf{Cov}\left[\theta_t, Y_t\right]}{\mathbf{Var}\left[Y_t\right]} \sum_{\ell=1}^{L} \alpha_{1,\ell} \int_0^\infty e^{-rt} V_\ell(t)\, \mathrm{d}t. \tag{OA.90}$$

Let

$$\alpha_k = \frac{\sum_{\ell=1}^{L} \alpha_{1,\ell}\alpha_{k,\ell}}{\sum_{\ell=1}^{L} \alpha_{1,\ell}}.$$

Maximizing (OA.90) is the same as maximizing

$$\frac{\mathbf{Cov}\left[\theta_t, \widetilde{Y}_t\right]}{\mathbf{Var}\left[\widetilde{Y}_t\right]} \int_0^\infty e^{-rt} \widetilde{V}(t)\, \mathrm{d}t, \tag{OA.91}$$

over $\{u_k\}_k$, where

$$\widetilde{Y}_t \coloneqq \sum_{k=1}^{K} \int_{s \le t} u_k(t-s)\, \mathrm{d}\widetilde{S}_{k,s},$$

and

$$\widetilde{V}(t) \coloneqq \sum_{k=1}^{K} \alpha_k u_k(t).$$

Note that the objective (OA.91) is the equilibrium marginal cost of the fictitious setting, under the confidential information structure generated by the rating process $\widetilde{Y}$.

Thus, the linear filter of the optimal confidential rating for the original setting is the same as the linear filter of the optimal confidential rating for the fictitious setting.

Part V

Complements for Appendix B

OA.13 Proof of Proposition B.1

If Y is a belief for a confidential information structure $\mathcal{M}$, then $Y_t = \mu_t$, with $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{M}_t] = \mathbf{E}^*[\theta_t \mid \mu_t]$, the second equality follows from the law of iterated expectations. As the pair of variables (θ_t, Y_t) is Gaussian, we apply the projection formulas to get

$$\mathbf{E}^*[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[\theta_t, Y_t]}{\mathbf{Var}[Y_t]}(Y_t - \mathbf{E}^*[Y_t]),$$

and thus $\mathbf{Cov}[\theta_t, Y_t] = \mathbf{Var}[Y_t]$ and $\mathbf{E}^*[Y_t] = 0$.

Conversely, if $\mathbf{Cov}[\theta_t, Y_t] = \mathbf{Var}[Y_t]$ and $\mathbf{E}^*[Y_t] = 0$, then the projection formula implies $\mathbf{E}^*[\theta_t \mid Y_t] = Y_t$. If $Y_t = \mathbf{E}^*[\theta_t \mid Y_t]$, then Y is the belief μ for the confidential information structure induced by Y, $\mathcal{M}_t = \sigma(Y_t)$.

OA.14 Proof of Lemma B.2

We note that the concept of equilibrium, presented in Definition 3.1, depends on the worker's information only through the set of worker strategies, and so remains valid without modification. The same observation applies to the proofs of Proposition 3.4 and Lemma B.3, which does not depend on the specific information available to the worker, because the optimal effort level specified by the equilibrium turns out to be independent of the history of realizations, and so independent of the worker's information.

OA.15 Proof of Proposition B.5

In this section, $\mathcal{R} = \{\mathcal{R}_t\}_{t\geq 0}$ is the natural augmented filtration generated by X, and let $\mathcal{F}$ be the natural augmented filtration generated by the pair (θ, X).

Note that if Y is as in the theorem statement, Y is a Gaussian process as well. Let assume, without loss, that the worker exerts zero effort, and that $\mathbf{E}[Y_t] = 0$ for every t. In this case,

$$X_t = Z_{1,t} + \int_0^t \theta_s \, \mathrm{d}t.$$

The proof makes use of the following lemma.

Lemma OA.1 *Fix $T > 0$ and let ξ be a square-integrable predictable process adapted to $\mathcal{R}$ defined over $[0, T]$. Suppose that there exists a function $f(s, t)$ defined on $\{(s, t) \in [0, T]^2 :$*

$s \leq t\}$, continuously differentiable in t, that satisfies the uniform Lipschitz continuity condition

$$\left|\frac{\partial f}{\partial t}(x,z) - \frac{\partial f}{\partial t}(y,z)\right| \leq M|x-y|,$$

for some M and all triples (x,y,z), and such that for every $s < t$,

$$\int_s^t \mathbf{E}[\xi_x \mid \mathcal{R}_s]\,\mathrm{d}x = f(s,t).$$

Then,

$$\mathbf{E}\left[\int_0^T (\xi_x - \partial f(x,x)/\partial t)^2\,\mathrm{d}x\right] = 0.$$

Proof of Lemma OA.1. Let us consider a sequence $\{\xi^k\}_k$ of simple processes (as defined, for example, in Karatzas and Shreve (1991), Chapter 3), such that

$$\mathbf{E}\left[\int_0^T (\xi_x^k - \xi_x)^2\,\mathrm{d}x\right] \to 0.$$

That ξ is square integrable and progressively measurable guarantees existence of such a sequence.

For every integer $k \geq 0$, let $\{(s_i^k, t_i^k]\}_i$ be a subdivision of the interval $(0,T]$ such that

$$\lim_{k\to+\infty} \max_i |t_i^k - s_i^k| = 0,$$

and

$$\xi_x^k = \mathbf{E}\left[\xi_x^k \mid \mathcal{R}_{s_i^k}\right].$$

The Cauchy-Schwarz inequality implies

$$\begin{aligned}
\frac{1}{5}\mathbf{E}\left[\int_0^T \left(\xi_x - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] &= \frac{1}{5}\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \frac{\partial f}{\partial t}(x,x)\right)^2 \mathrm{d}x\right] \\
&\leq \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x^k - \mathbf{E}\left[\xi_x^k \mid \mathcal{R}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] \\
&\quad + \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x^k \mid \mathcal{R}_{s_i^k}\right] - \mathbf{E}\left[\xi_x \mid \mathcal{R}_{s_i^k}\right]\right)^2 \mathrm{d}x\right]
\end{aligned}$$

$$+\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \mid \mathcal{R}_{s_i^k}\right] - \frac{\partial f}{\partial t}(s_i^k, x)\right)^2 \mathrm{d}x\right]$$
$$+\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\frac{\partial f}{\partial t}(s_i^k, x) - \frac{\partial f}{\partial t}(x, x)\right)^2 \mathrm{d}x\right].$$

Let us show that each term converges to zero. First, by choice of the sequence of simple processes, we immediately get

$$\lim_{k\to+\infty} \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right] = 0.$$

Also, by choice of the subdivisions, we immediately get

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x^k - \mathbf{E}\left[\xi_x^k \mid \mathcal{R}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] = 0.$$

Then, the hypothesis that, for every $s < t$, we have

$$\int_s^t \mathbf{E}[\xi_x \mid \mathcal{R}_s] \,\mathrm{d}x = f(s,t)$$

implies that $t \mapsto f(s,t)$ is absolutely continuous when $t \geq s$, and thus for almost every $\omega \in \Omega$ and almost every $t \geq s$,

$$\mathbf{E}[\xi_t \mid \mathcal{R}_s] \,\mathrm{d}x = \frac{\partial f}{\partial t}(s,t).$$

In turn, this last equality implies

$$\lim_{k\to+\infty} \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{R}_{s_i^k}\right] - \frac{\partial f}{\partial t}(s_i^k, x)\right)^2 \mathrm{d}x\right] = 0.$$

Next, we observe that by Jensen's inequality,

$$\left(\mathbf{E}\left[\xi_x - \xi_x^k \,\middle|\, \mathcal{R}_{s_i^k}\right]\right)^2 \leq \mathbf{E}\left[\left(\xi_x - \xi_x^k\right)^2 \,\middle|\, \mathcal{R}_{s_i^k}\right],$$

which after integration implies

$$\int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{R}_{s_i^k}\right] - \mathbf{E}\left[\xi_x^k \,\middle|\, \mathcal{R}_{s_i^k}\right]\right)^2 \mathrm{d}x \leq \int_{s_i^k}^{t_i^k} \mathbf{E}\left[\left(\xi_x - \xi_x^k\right)^2 \,\middle|\, \mathcal{R}_{s_i^k}\right] \mathrm{d}x,$$

and so, by the law of iterated expectations,

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\mathbf{E}\left[\xi_x \,\middle|\, \mathcal{R}_{s_i^k}\right] - \mathbf{E}\left[\xi_x^k \,\middle|\, \mathcal{R}_{s_i^k}\right]\right)^2 \mathrm{d}x\right] \leq \mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\xi_x - \xi_x^k\right)^2 \mathrm{d}x\right].$$

Finally,

$$\mathbf{E}\left[\sum_i \int_{s_i^k}^{t_i^k} \left(\frac{\partial f}{\partial t}(s_i^k, x) - \frac{\partial f}{\partial t}(x, x)\right)^2 \mathrm{d}x\right] \leq M \sum_i |t_i^k - s_i^k|^3 \to 0.$$

Hence, each term converges to zero, which naturally implies

$$\mathbf{E}\left[\int_0^T \left(\xi_x - \frac{\partial f}{\partial t}(x, x)\right)^2 \mathrm{d}x\right] = 0,$$

and concludes the proof. ■

We now return to the main proof of Proposition B.5.

Let $T > 0$. In most of the proof, we work on the finite horizon $[0, T]$.

Let the process L be defined by $L_t = -\int_0^t \theta_s \, \mathrm{d}Z_{1,s}$, and H be the Doléans-Dade exponential of L, *i.e.*,

$$H_t = \exp\left(L_t - \frac{1}{2}\int_0^t \theta_s^2 \, \mathrm{d}s\right).$$

The Novikov condition is satisfied on $[0, T]$ (*e.g.*, Corollary 3.5.13 of Karatzas and Shreve (1991)).Therefore, H is a martingale density process for some probability measure Q equivalent to P on $\mathcal{F}_T$. By the Girsanov-Cameron-Martin Theorem (*e.g.*, Theorem 3.5.1 of Karatzas and Shreve (1991)),

$$W_t + \int_0^t \theta_s \, \mathrm{d}s \qquad (= X_t)$$

is a standard Brownian motion on $\mathcal{F}$, for $t \in [0, T]$, with respect to Q.

We observe that Y_T is a square-integrable random variable measurable with respect to $\mathcal{R}_T$. In addition, under Q, X is a standard Brownian motion. Thus, invoking the Martingale Representation Theorem under Q (*e.g.*, see Karatzas and Shreve (1991), Section 3.4), there exists a square-integrable predictable process $\xi = \{\xi_t\}_{t \in [0,T]}$ adapted to $\mathcal{R}$ such that, almost surely with respect to Q,

$$Y_T = \mathbf{E}^Q[Y_T] + \int_0^T \xi_t \, \mathrm{d}X_t, \tag{OA.92}$$

where $\mathbf{E}^Q$ denotes the expectation operator under Q. Since P and Q are equivalent probability measures, we immediately have that (OA.92) also holds almost surely under P.

Then, using $\mathbf{E}[Y_T] = 0$ and $\mathbf{E}[\mathrm{d}X_t] = 0$, we get

$$Y_T = \int_0^T \xi_t \, \mathrm{d}X_t$$

(in the almost sure sense).

The conditions of the proposition statement imply that

$$\begin{aligned}\mathbf{Cov}\left[Y_T, \int_t^{t+\tau} (\mathrm{d}X_x - \theta_x \, \mathrm{d}x) \,\middle|\, \mathcal{R}_t\right] &= \mathbf{Cov}\left[Y_T, \int_t^{t+\tau} \mathrm{d}Z_{1,x} \,\middle|\, \mathcal{R}_t\right] \\ &= \int_t^{t+\tau} \mathbf{E}\left[\xi_x \mid \mathcal{R}_t\right] \mathrm{d}x,\end{aligned}$$

where the equality is obtained using Itô's isometry, and uses the fact that ξ is adapted to $\mathcal{R}$ and that Z_1 remains a standard Brownian motion in the filtration $\mathcal{R}$. We conclude by application of Lemma OA.1.

OA.16 Proof of Proposition B.6

Here, we prove that the optimal exponential smoothing system dominates any moving window system under confidential (exclusive) information structures, in the context of two signals, $S_1 = X$ (the output) and $S_2 = S$.[6] For simplicity, given that the parameters of the output process $\alpha_1 = \beta_1$ are normalized to 1, we simply write α for α_2, β for β_2, σ for σ_1 and ϵ for σ_2.

Recall that a moving window system is defined, up to an additive constant, by a two-parameter rating process

$$Y_t = \int_{t-\tau}^{t} c \, \mathrm{d}X_j + (1-c) \, \mathrm{d}S_j,$$

at time t, which is the market information $\mathcal{M}_t$. The parameters are $\tau > 0$, the size of the moving window, and c, the relative weight put on the output.

Straightforward calculations yield that, in the stationary equilibrium, equilibrium effort a^{mw}, if positive, satisfies

$$c'(a^{\mathrm{mw}}) = \gamma^2(\beta(1-c)+c)(c+(1-c)\alpha)\frac{1-e^{-r\tau}}{r}\rho^{\mathrm{mw}}, \qquad \text{(OA.93)}$$

[6] We suspect that the result generalizes to more signals, but have not investigated this claim.

with

$$\rho^{\mathrm{mw}} := \frac{(1-e^{-\tau})}{2\left(c^2\tau\sigma^2+\gamma^2\left(\tau+e^{-\tau}-1\right)(\beta-\beta c+c)^2+(c-1)^2\tau\epsilon^2\right)}.$$

Instead, an exponential smoothing system is defined (up to an additive constant) by a two-parameter rating process

$$Y_t = \int_{j\le t} e^{-\lambda(t-j)}(c\,\mathrm{d}X_j + (1-c)\,\mathrm{d}S_j),$$

at time t, which is the market information $\mathcal{M}_t$. Here, the parameters are $\lambda > 0$, the coefficient of smoothing, and c, as before, the relative weight put on the output.

It is readily verified that, in the stationary equilibrium, equilibrium effort a^{es}, if positive, satisfies

$$c'(a^{\mathrm{es}}) = \gamma^2(\beta(1-c)+c)(c+(1-c)\alpha)\frac{\rho^{\mathrm{es}}}{r+\lambda}, \tag{OA.94}$$

with

$$\rho^{\mathrm{es}} := \frac{\lambda}{(\lambda+1)\left(c^2\sigma^2+\frac{\gamma^2(\beta-\beta c+c)^2}{\lambda+1}+(c-1)^2\epsilon^2\right)}.$$

Our objective is to prove that the maximum over (c,λ) of the right-hand side of (OA.94) exceeds the maximum over (c,τ) of the right-hand side of (OA.93).

We will prove this pointwise in c. First, note that if $(\beta(1-c)+c)(c+(1-c)\alpha)\le 0$, there is nothing to show. So we are left with showing that

$$\sup_{\lambda>0}\min_{\tau\ge 0} f(c,\tau,\lambda)\ge 0,$$

where

$$f(c,\tau,\lambda) := \frac{\rho^{\mathrm{es}}}{r+\lambda} - \frac{1-e^{-r\tau}}{r}\rho^{\mathrm{mw}}.$$

Let

$$g = \frac{\gamma^2(\beta(1-c)+c)^2}{c^2\sigma^2+(c-1)^2\epsilon^2},$$

which is clearly nonnegative. Computing f, we obtain

$$f(c,\tau,\lambda) = \frac{2\lambda r\left(e^\tau(\tau+(\tau-1)g)+g\right)e^{\tau r} - (e^\tau-1)\left(g+\lambda+1\right)\left(e^{\tau r}-1\right)(\lambda+r)}{2r\left(e^\tau(\tau+(\tau-1)g)+g\right)(g+\lambda+1)e^{\tau r}(\lambda+r)\left(c^2\sigma^2+(c-1)^2\epsilon^2\right)}.$$

The elementary inequality

$$e^\tau(\tau-1)+1\ge 0$$

implies that

$$e^{\tau}(\tau + (\tau - 1)g) + g \geq 0,$$

and so the denominator of f is positive. We note that the numerator is quadratic in λ, concave, with

$$f(c, \tau, 0) = (1 - e^{\tau})\,(g + 1)r\,(e^{\tau r} - 1) \leq 0,$$

and it is readily verified that also $\partial f(c, \tau, 0)/\partial\lambda \geq 0$. So it suffices to show the discriminant $\widetilde{\tau}$ is positive. A simple calculation shows that this discriminant is quadratic and convex in g, as

$$\frac{\mathrm{d}^2\widetilde{\tau}}{\mathrm{d}g^2} = 2\left(e^{\tau} + e^{\tau r}\left(e^{\tau}(2(\tau - 1)r - 1) + 2r + 1\right) - 1\right)^2 \geq 0.$$

Evaluating the minimum of $\widetilde{\tau}$ with respect to g, we get

$$\min_{g} \widetilde{\tau} = (x - 1)^3 x^r\,(x^r - 1)^2\left((1 - x)\left((r + 1)x^r + r - 1\right) + rx\,(x^r + 1)\ln x\right),$$

where $x \coloneqq e^{\tau} \geq 1$. Hence it remains to argue that

$$(1 - x)\left((r + 1)x^r + r - 1\right) + rx\,(x^r + 1)\ln x \geq 0,$$

or equivalently,

$$\frac{(1 - x)\left((r + 1)x^r + r - 1\right)}{x\,(x^r + 1)} + r\ln x \geq 0.$$

This is clearly true for $x = 1$, and the derivative of the left-hand side with respect to x is

$$\frac{r(x - 1)\left(x^{2r} + 1\right) + 1 - x^{2r}}{x^2\,(x^r + 1)^2},$$

which is positive because

$$\begin{aligned}
r(x - 1)\left(x^{2r} + 1\right) + 1 - x^{2r} &= r(y^{\frac{1}{2r}} - 1)\,(y + 1) + 1 - y \\
&\geq r\left(1 + \frac{1}{2r}(y - 1) - 1\right)(y + 1) + 1 - y \\
&= \frac{1}{2}(y - 1)^2,
\end{aligned}$$

where $y \coloneqq x^{2r}$, and we use Bernoulli's inequality in the second line.

OA.17 Proof of Lemma B.7

We note that θ_t and μ_t are jointly normal, and as μ_t is the market belief, $\mathbf{Cov}[\theta_t, \mu_t] = \mathbf{Var}[\mu_t]$ by Proposition B.1, so applying the projection formulas, we obtain

$$\mathbf{Var}[\theta_t \mid \mu_t] = \mathbf{Var}[\theta_t] - \frac{\mathbf{Cov}[\theta_t, \mu_t]^2}{\mathbf{Var}[\mu_t]} = \frac{\gamma^2}{2} - \mathbf{Var}[\mu_t].$$

OA.18 Proof of Proposition B.9

If Y is a belief for a public information structure $\mathcal{M}$, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{M}_t] = \mathbf{E}^*[\theta_t \mid \{\mu_s\}_{s\leq t}]$. The second equality follows from the law of iterated expectations, using that $\mathcal{M}$ is a filtration and thus $\mathcal{M}_t$ includes all information about $\{\mu_s\}_{s\leq t}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{Y_s\}_{s\leq t}]$, then Y is the belief μ for the public information structure that corresponds to the filtration generated by Y, $\mathcal{M}_t = \sigma(\{Y_s\}_{s\leq t})$. This proves the second part of the proposition.

We now proceed to the proof of the first part. Note that the correlation between θ_t and $\theta_{t+\tau}$ satisfies

$$\mathbf{Corr}[\theta_t, \theta_{t+\tau}] = \frac{\mathbf{Cov}[\theta_t, \theta_{t+\tau}]}{\sqrt{\mathbf{Var}[\theta_t]}\sqrt{\mathbf{Var}[\theta_{t+\tau}]}} = e^{-\tau},$$

since, as θ is a stationary Ornstein-Uhlenbeck process with reversion rate 1 and scale γ,

$$\mathbf{Cov}[\theta_t, \theta_{t+\tau}] = \frac{\gamma^2}{2} e^{-\tau}, \text{ and } \mathbf{Var}[\theta_t] = \mathbf{Var}[\theta_{t+\tau}] = \frac{\gamma^2}{2}.$$

Let μ be the market belief process induced by some public information structure $\mathcal{M}$. Assume μ is a linear and stationary rating process. We have $\mathbf{E}^*[\mu_t] = \mathbf{E}^*[\theta_t] = 0$. As $\mathcal{M}$ is also a confidential information structure, μ is also a belief for a confidential information structure. Conditionally on μ_t, the random variable θ_t is then independent from every $\mu_{t-\tau}$, $\tau \geq 0$, because μ_t carries all relevant information about θ_t. Thus, $\mathbf{Cov}[\theta_t, \mu_{t-\tau} \mid \mu_t] = 0$. Let $\tau \geq 0$. The projection formulas for jointly Gaussian random variables yield

$$\mathbf{Cov}[\theta_t, \mu_{t-\tau} \mid \mu_t] = \mathbf{Cov}[\theta_t, \mu_{t-\tau}] - \frac{\mathbf{Cov}[\theta_t, \mu_t]\,\mathbf{Cov}[\mu_{t-\tau}, \mu_t]}{\mathbf{Var}[\mu_t]}.$$

Hence,

$$\mathbf{Cov}[\mu_{t-\tau}, \mu_t] = \mathbf{Var}[\mu_t]\frac{\mathbf{Cov}[\theta_t, \mu_{t-\tau}]}{\mathbf{Cov}[\theta_t, \mu_t]} = \mathbf{Var}[\mu_{t-\tau}]\frac{\mathbf{Cov}[\theta_t, \mu_{t-\tau}]}{\mathbf{Cov}[\theta_{t-\tau}, \mu_{t-\tau}]}, \tag{OA.95}$$

using stationarity of (μ, θ). Writing μ_t as

$$\mu_t = \sum_{k=1}^{K} \int_{s \leq t} u_k^{\mu}(t-s)[\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s]$$

for some linear filter $\{u_1^{\mu}, \ldots, u_K^{\mu}\}$, we get, using $\mathbf{Cov}[\theta_t, \theta_{t-\tau}] = \gamma^2 e^{-\tau}/2$,

$$\begin{aligned} \mathbf{Cov}[\mu_{t-\tau}, \theta_{t-\tau}] &= \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^{\infty} u_k^{\mu}(s) e^{-s} \,\mathrm{d}s, \text{ and} \\ \mathbf{Cov}[\mu_{t-\tau}, \theta_t] &= \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^{\infty} u_k^{\mu}(s) e^{-(\tau+s)} \,\mathrm{d}s = e^{-\tau}\, \mathbf{Cov}[\mu_{t-\tau}, \theta_{t-\tau}]. \end{aligned}$$

Plugging these last two expressions into (OA.95), we have

$$\mathbf{Cov}[\mu_t, \mu_{t+\tau}] = \mathbf{Cov}[\mu_{t-\tau}, \mu_t] = \mathbf{Var}[\mu_{t-\tau}] e^{-\tau} = \mathbf{Var}[\mu_t] e^{-\tau}.$$

Now, we prove the converse. Let Y be a linear stationary rating process that is a belief for a confidential information structure, and satisfies

$$\mathbf{Cov}[Y_{t+\tau}, Y_t] = \mathbf{Var}[Y_t] e^{-\tau},$$

for every $\tau \geq 0$. Writing Y_t explicitely as

$$Y_t = \sum_{k=1}^{K} \int_{s \leq t} u_k^{Y}(t-s) \left(\mathrm{d}S_{k,s} - \alpha_k A_s^* \,\mathrm{d}s\right),$$

for the linear filter $\{u_1^{Y}, \ldots, u_K^{Y}\}$, we get, as above,

$$\mathbf{Cov}[Y_{t-\tau}, \theta_t] = e^{-\tau}\, \mathbf{Cov}[Y_{t-\tau}, \theta_{t-\tau}] = e^{-\tau}\, \mathbf{Cov}[Y_t, \theta_t],$$

using the stationarity of (Y, θ), and we have by assumption on Y that

$$e^{-\tau} = \frac{\mathbf{Cov}[Y_t, Y_{t-\tau}]}{\mathbf{Var}[Y_{t-\tau}]} = \frac{\mathbf{Cov}[Y_t, Y_{t-\tau}]}{\mathbf{Var}[Y_t]}.$$

Therefore,

$$\mathbf{Cov}[\theta_t, Y_{t-\tau} \mid Y_t] = \mathbf{Cov}[\theta_t, Y_{t-\tau}] - \frac{\mathbf{Cov}[\theta_t, Y_t]\, \mathbf{Cov}[Y_{t-\tau}, Y_t]}{\mathbf{Var}[Y_t]} = 0.$$

As θ and Y are jointly Gaussian, it implies that θ_t and $Y_{t-\tau}$ are independent conditionally on Y_t for every $\tau \geq 0$, so the market belief associated with the public information structure that is the filtration generated by Y satisfies

$$\mathbf{E}^* \left[\theta_t \mid \{Y_s\}_{s \le t}\right] = \mathbf{E}^* \left[\theta_t \mid Y_t\right] = Y_t.$$

The conclusion follows from the second part of the proposition, proved above.

OA.19 Proof of Proposition B.11

The proposition has two parts. We begin by the proof of the first part of the proposition.

- If Y is a belief for a confidential information structure $\mathcal{M}$, with nonexclusive signals $S_1, \ldots, S_{K_0}$, then $Y_t = \mu_t$, where, by definition,

  $$\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{M}_t] = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, \mu_t].$$

  Note that the second equality follows from the law of iterated expectations, using that $\mathcal{M}_t$ includes all information about $\{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, Y_t]$, then Y is the belief μ for the confidential information structure induced by rating Y and the history of signals $S_1, \ldots, S_{K_0}$.

- If Y is a belief for a public information structure structure $\mathcal{M}$, with nonexclusive signals $S_1, \ldots, S_{K_0}$, then $Y_t = \mu_t$, where, by definition,

  $$\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{M}_t] = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, \{\mu_s\}_{s \le t}],$$

  using that $\mathcal{M}$ is a filtration and includes all information about $\{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t \mid \{S_{k,s}\}_{s \le t, k=1,\ldots,K_0}, \{Y_s\}_{s \le t}]$, then Y is the belief μ for the public information structure that is the filtration generated by Y and the signals $S_1, \ldots, S_{K_0}$.

We now proceed to the proof of the second part. Let Y be a stationary linear rating that is the belief of a confidential or public information structure with nonexclusive signals $S_1, \ldots, S_{K_0}$. At a given time t, conditionally on Y_t, the random variable θ_t is independent from all past nonexclusive signals $S_{k,s}$, $s \le t$. That is, $\mathbf{Cov}[\theta_t, S_{k,s} \mid Y_t] = 0$ for every $k = 1, \ldots, K_0$, as the market belief Y_t carries by assumption all relevant information about θ_t that is already contained in the nonexclusive signals. Hence, for every $k = 1, \ldots, K_0$, every t and every $\tau \ge 0$,

$$\mathbf{Cov}[\theta_{t+\tau}, S_{k,t} \mid Y_{t+\tau}] = 0. \tag{OA.96}$$

By the projection formulas for jointly Gaussian random variables,

$$\mathbf{Cov}[\theta_{t+\tau}, S_{k,t} \mid Y_{t+\tau}] = \mathbf{Cov}[\theta_{t+\tau}, S_{k,t}] - \frac{\mathbf{Cov}[\theta_{t+\tau}, Y_{t+\tau}]\,\mathbf{Cov}[S_{k,t}, Y_{t+\tau}]}{\mathbf{Var}[Y_{t+\tau}]}.$$

By Proposition B.1, $\mathbf{Cov}[\theta_{t+\tau}, Y_{t+\tau} = \mathbf{Var}[Y_{t+\tau}]$. Thus,

$$\mathbf{Cov}[\theta_{t+\tau}, Y_{t+\tau}] = \mathbf{Cov}[\theta_{t+\tau}, S_{k,t}] \; \forall k = 1, \ldots, K_0, \forall t, \forall \tau \geq 0. \tag{OA.97}$$

Conversely, suppose Y is a rating process that is a market belief associated with a confidential structure which satisfies (OA.97). Reversing the argument above, the equality (OA.96) must hold. As all random variables involved are jointly Gaussian, given any time t, such equality implies that conditionally on Y_t, the random variable θ_t is independent from all past nonexclusive signals $S_{k,s}$, $s \leq t$, $k = 1, \ldots, k_{K_0}$. Hence, by Proposition B.1

$$\mathbf{E}^* \left[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, Y_t\right] = \mathbf{E}^*[\theta_t \mid Y_t] = Y_t,$$

and it follows from the first part of the proposition, proven at the beginning of this section, that Y is a market belief for a confidential information structure with nonexclusive signals $S_1, \ldots, S_{K_0}$. If instead Y is a market belief associated with a public structure, then

$$\mathbf{E}^* \left[\theta_t \mid \{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}, \{Y_s\}_{s \leq t}\right] = \mathbf{E}^*[\theta_t \mid Y_t] = Y_t,$$

and applying again the first part of the proposition, Y is a market belief for a public information structure with nonexclusive signals $S_1, \ldots, S_{K_0}$.

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Part VI
Ancillary Mathematical Results

In this part, we derive the first-order conditions for the type of control problems considered in the paper by adapting the classical Euler-Lagrange conditions.

Let $N, M, K, L,$ be positive integers. For $\ell = 1, \ldots, L$, let $F^\ell : \mathbf{R}_+^N \to \mathbf{R}$, and $G^\ell : \mathbf{R}^{K\times M} \to \mathbf{R}$, where every G^ℓ can be written

$$G^\ell((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M})) = y_{k,i} y_{k',i'},$$

for some k, k', i, i'. In other words, letting

$$F(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_M) = \sum_{\ell=1}^{L} F^\ell(\mathbf{x}) G^\ell(\mathbf{y}_1, \ldots, \mathbf{y}_M),$$

we have that $F(\mathbf{x}, \cdot)$ is a quadratic form, and $F^\ell(\mathbf{x})$ are the coefficients.

For every $i = 1, \ldots, M$, let $\phi_i : \mathbf{R}_+^N \to \mathbf{R}_+$ be a (possibly shifted) projection, in the following sense: $\phi_i((x_1; \ldots; x_N)) = x_j + \delta$ for some j and some $\delta \geq 0$. Let $\mathcal{U}$ be the space of measurable functions $\mathbf{u} : \mathbf{R}_+ \to \mathbf{R}^K$ that are continuous, integrable and square integrable.

Define $G^\ell_{k,i}((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M}))$ as

$$\frac{\partial G((y_{1,1}; \ldots; y_{K,1}), \ldots, (y_{1,M}; \ldots; y_{K,M}))}{\partial y_{k,i}},$$

and let

$$F_{k,i}(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_M) = \sum_{\ell=1}^{L} F^\ell(\mathbf{x}) G^\ell_{k,i}(\mathbf{y}_1, \ldots, \mathbf{y}_M).$$

We consider the problem of maximizing

$$\int_{\mathbf{R}_+^N} F(\mathbf{x}, \mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x}, \tag{OA.98}$$

over control functions $\mathbf{u} \in \mathcal{U}$.

We make the following assumptions:

1. For every ℓ, every $u \in \mathcal{U}$, $\mathbf{x} \mapsto F^\ell(\mathbf{x}) G^\ell(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))$ is integrable on $\mathbf{R}_+^N$.

2. For every ℓ, i, k, $\mathbf{x} \mapsto F^\ell(\mathbf{x}) G^\ell_{k,i}(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))$ is integrable on $\mathbf{R}_+^N \cap \{\phi_i = t\}$ for every t.

3. The map

$$t \mapsto \int_{\mathbf{R}_+^N \cap \{\phi_i = t\}} F^\ell(\mathbf{x}) G^\ell_{k,i}(\mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x}$$

is piecewise continuous, where the integral is taken with respect to the Lebesgue measure on $\mathbf{R}_+^N \cap \{\phi_i = t\}$.

Compared to standard problems of calculus of variations (see, for example, Burns (2014), Chapter 3), this optimization problem involves delayed terms and integrals over a domain whose dimension is unrelated to the dimension of the control. The classical Euler-Lagrange equations do not hold. However, the argument can be adapted to yield the following first-order condition.

Proposition OA.1 *Assume the control function* $\mathbf{u}^* \in \mathcal{U}$ *maximizes* (OA.98)*. Then, for every* k *and every* t*,*

$$\sum_{i=1}^{M} \int_{\mathbf{R}_+^N \cap \{\phi_i = t\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x} = 0.$$

Proof. For a control function $\mathbf{u} \in \mathcal{U}$, let

$$J(\mathbf{u}) := \int_{\mathbf{R}_+^N} F(\mathbf{x}, \mathbf{u}(\phi_1(\mathbf{x})), \mathbf{u}(\phi_2(\mathbf{x})), \ldots, \mathbf{u}(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x},$$

and assume $J(\mathbf{u})$ is maximized for $\mathbf{u} = \mathbf{u}^*$.

The proof relies on classical variational arguments. Fix k and let $\mathbf{v} : \mathbf{R}_+ \to \mathbf{R}^K$, where we write $\mathbf{v} = (v_1, \ldots, v_K)$ and where $v_{k'} = 0$ for $k' \neq k$, and assume v_k is continuous with bounded support. Let $j(\epsilon) = J(\mathbf{u}^* + \epsilon \mathbf{v})$. Differentiating under the integral sign (see, for example, Theorem 6.28 of Klenke (2014)), we get

$$j'(0) = \int_{\mathbf{R}_+^N} \sum_{i=1}^{M} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x}))) v_k(\phi_i(\mathbf{x}))\, \mathrm{d}\mathbf{x}.$$

We observe that j is maximized at $\epsilon = 0$, and so $j'(0) = 0$.

Suppose by contradiction that, for some t,

$$\sum_{i=1}^{M} \int_{\mathbf{R}_+^N \cap \{\phi_i = t\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x}$$

is nonzero—for example, positive. The sum is piecewise continuous with respect to t, and so by continuity,

$$\sum_{i=1}^{M} \int_{\mathbf{R}_{+}^{N} \cap \{\phi_i = t'\}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x})))\, \mathrm{d}\mathbf{x}$$

is positive for t' on an interval to the left or the right of t. Let I_t be such an interval, and let v_k be a function that is zero outside of I_t and that is positive inside I_t. Then

$$\begin{aligned} 0 < \int_{t' \in I_t} \sum_{i=1}^{M} \int_{\mathbf{R}_{+}^{N} \cap \{\phi_i = t'\}} & F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x})))v_k(t')\, \mathrm{d}\mathbf{x}\, \mathrm{d}t' \\ &= \sum_{i=1}^{M} \int_{\mathbf{R}_{+}^{N}} F_{k,i}(\mathbf{x}, \mathbf{u}^*(\phi_1(\mathbf{x})), \mathbf{u}^*(\phi_2(\mathbf{x})), \ldots, \mathbf{u}^*(\phi_M(\mathbf{x})))v_k(\phi_i(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \end{aligned}$$

which contradicts $j'(0) = 0$. ■

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Part VII

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