Discrete Stationary Model for "Motivational Ratings"

Johannes Hörner and Nicolas S. Lambert

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This document describes the discrete stationary model whose optimal rating is derived in the Mathematica© file "StationaryDiscreteSolution.nb." The model is based directly on the discrete-time model of Section 2 of the paper.

Time is indexed by $t \in \mathbf{Z}$: the origin of time is $-\infty$, and the horizon is infinite. This origin of time enables us to have, at any given instant, an infinite history of outputs, which in turn permits a proper formulatation of stationary linear ratings.

In period t, the worker generates output

$$X_t = A_t + \theta_t + \sigma\epsilon_t,$$

where ϵ_t is drawn independently from the Gaussian distribution with mean 0 and variance 1, A_t is the worker's effort in period t, and θ_t is the worker's ability in this period. Here, θ is defined as the stationary mean reverting process that satisfies, for all $t \in \mathbf{Z}$, $\theta_t = \rho\theta_{t-1} + \gamma\epsilon'_t$, with ϵ'_t independently drawn from the Gaussian distribution with mean 0 and variance 1. This definition implies that θ_t follows the Gaussian distribution with mean 0 and variance

$$\frac{\gamma^2}{1-\rho^2}$$

and that, in addition,

$$\mathrm{Cov}[\theta_{t-i}, \theta_{t-j}] = \rho^{|j-i|}\frac{\gamma^2}{1-\rho^2},$$

we let Σ_{ij} be equal to that covariance.

The rater observes the history of past outputs and provides to the market a rating Y_t at the beginning of every period t, which is the only information the market has in this period. At the beginning of period t, the market forms belief $\mu_t = \mathrm{E}[\theta_t \mid Y_t]$ about the mean of the worker's ability.

The focus is on stationary linear ratings. A stationary linear rating is a process Y that satisfies, for all $t \in \mathbf{Z}$ and up to a additive constant, the equality

$$Y_t = \sum_{k \geq 1} u_k X_{t-k},$$

with $\{u_k\}_k$ a sequence of real numbers such that $\sum_k u_k^2 < \infty$.

The setting is stationary and the market's expectation about the worker's effort is a constant. The worker's discount factor is δ, and as in Section 2 of the paper, the worker's cost of effort is quadratic and so the worker chooses the constant effort A that maximizes

$$\sum_{t \geq 1} \delta^{t-1} \left(\mathrm{E}[\mu_t] - \frac{A^2}{2} \right).$$

If the rating is equal to the market belief, i.e., if $Y_t = \mu_t$, then the first-order condition for the worker is to choose, at every period, the constant effort level

$$A = \sum_{k \geq 1} \delta^k u_k.$$

Without loss, we restrict attention to ratings that satisfy the normalizing condition $\mathrm{Cov}[\theta_t, Y_t] = \mathrm{Var}[Y_t]$, which implies that the rating is equal to the market belief (up to an additive constant). Note that we have

$$\mathrm{Cov}[\theta_t, Y_t] = \frac{\gamma^2}{1 - \rho^2} \sum_{k \geq 1} u_k \rho^k,$$

and

$$\begin{aligned} \mathrm{Var}[Y_t] &= \sum_{i \geq 1} \sum_{j \geq 1} u_j u_j \,\mathrm{Cov}[X_{t-i}, X_{t-j}] \\ &= \sigma^2 \sum_{k \geq 1} u_k^2 + \sum_{i \geq 1} \sum_{j \geq 1} u_j u_j \Sigma_{ij}. \end{aligned}$$

Let us say that a stationary linear rating Y is optimal if it maximizes

$$\sum_{k\geq 1} \delta^k u_k$$

subject to the normalizing condition $\mathrm{Cov}[\theta_t, Y_t] = \mathrm{Var}[Y_t]$, i.e.,

$$\sigma^2 \sum_{k\geq 1} u_k^2 + \sum_{i\geq 1}\sum_{j\geq 1} u_j u_j \Sigma_{ij} - \frac{\gamma^2}{1-\rho^2}\sum_{k\geq 1} u_k \rho^k = 0.$$

To solve this problem, we internalize the constraint and maximize, over $\{u_k\}_k$, the objective

$$\sum_{k\geq 1} \delta^k u_k + \lambda\sigma^2 \sum_{k\geq 1} u_k^2 + \lambda \sum_{i\geq 1}\sum_{j\geq 1} u_j u_j \Sigma_{ij} - \frac{\lambda\gamma^2}{1-\rho^2}\sum_{k\geq 1} u_k \rho^k,$$

where λ is the Lagrange multiplier associated with the normalization constraint. We assume a solution exists, and under this assumption, we obtain the unique solution by solving the equations associated to the first- and second-order conditions. The solution and its derivation is in the Mathematica© file "StationaryDiscreteSolution.nb."